A method based on EUTL-MM operators to multiple attribute group decision making under uncertain 2-tuple linguistic environment

Yi Liu [1,2] **, Jun Liu** [3] **, Ya Qin** [4]

[1] *National-Local Joint Engineering Laboratory of System Credibility Automatic Verification, Southwest Jiaotong University,*
No. 111 North Section 1 of the second ring road
Chengdu, Sichuan 610031, P. R. China

[2] *Data Recovery Key Laboratory of Sichuan Province, Neijiang Normal University,*
No. 705 of Dongtong Road
Neijiang, Sichuan 641000, P.R. China
E-mail: liuyiyl@126.com

[3] *School of Computing, Ulster University at Jordanstown Campus,*
Shore Road, Newtownabbey
Co. Antrim, Northern Ireland BT37 0QB, UK
E-mail: j.liu@ulster.ac.uk

[4] *School of Mathematics and Information Sciences, Neijiang Normal University,*
No. 705 of Dongtong Road
Neijiang, Sichuan 641000, P.R. China
E-mail: qinyaqy@126.com

Abstract

The present work is focused on multi-attribute group decision making (MAGDM) problems with the uncertain 2-tuple linguistic information based on new aggregation operator which can capture interrelationships of attributes among any number of attributes by a parameter vector P. To begin with, we present some new uncertain 2-tuple linguistic MM aggregation operators to deal with MAGDM problems with uncertain 2-tuple linguistic information, including the uncertain 2-tuple linguistic Muirhead mean (UTL-MM) operator, uncertain 2-tuple linguistic weighted Muirhead mean (UTL-WMM) operator. In addition, we extend UTL-WMM operator to a new uncertain 2-tuple linguistic weighted Muirhead mean (named EUTL-WMM) operators in order to deal with some decision making problems with uncertain 2-tuple linguistic information whose attribute values are expressed in uncertain 2-tuple linguistic information and attribute weights are also 2-tuple linguistic information. Whilst, the some properties of these new aggregation operators are obtained and some special cases are discussed. Moreover, we propose a new method to solve the MAGDM problems with uncertain 2-tuple linguistic information. Finally, we use an illustrative example to show the feasibility and validity of the new method by comparing with the other existing methods.

Keywords: Modified uncertain 2-tuple linguistic representation model, uncertain 2-tuple linguistic weighted Muirhead mean (named EUTL-WMM) operators, multi-attribute group decision making (MAGDM)

1. Introduction

There are many complicated or ill-defined problems are not to be amenable for expressions in conventional quantitative ways in the real world, so it is not always adequate to represent such problems by only numerical based modelling. Therefore, the decision makers (DMs) utilize linguistic descriptors to express their assessments on the uncertain knowledge when they encounter such problems. Many studies on using the linguistic variables to model the problems have been carried out and have applied successfully in different fields. In multi-attribute decision making (MADM) problems, the linguistic decision information needs to be aggregated by some proper methods in order to rank the given decision alternatives and then to get the best one. On basis of the concept of symbolic translation, Herrera ect. [1,?] proposed 2-tuple linguistic representation model which was characterized by a linguistic term and a numeric value. It has exact characteristic in linguistic information processing and can effectively avoid information distortion and losing which occur formerly in the linguistic information processing. The 2-tuple linguistic model has received more and more attention since its appearance. Some extensions of 2-tuple linguistic model have been developed, e. g. hesitant 2-tuple linguistic information model [3,4,5,6,7,8,9,10,11,12], intuitionistic 2-tuple linguistic information model [13,14,15,16]. Whilst, a variety of decision making methods based on 2-tuple linguistic model are also developed, for example, FLINSTONES [17], VIKOR method [18,19,20], novel approach for FMEA [21], Grey 2-tuple linguistic evaluation method [22], ELECTRE II [23], TOPSIS method [24] etc.

In the field of information fusion, information aggregation is an important research topic as it is a critical process of gathering relevant information from multiple sources. However, aggregation operator as a tool to aggregate relevant information has been focused and also used in many decision making problems. In linguistic decision making, many 2-tuple aggregation operators have been proposed for information aggregation. We divide these 2-tuple linguistic aggregation operators into following five categories after reviewing related work: (1) 2-tuple linguistic aggregation operators based on Choquet integral. For example, Yang and Chen proposed 2-tuple correlated averaging (TCA) operator and generalized 2-tuple correlated averaging (GTCA) operator based on Choquet integral and 2-tuple linguistic information in [25]. Merigo [26] presented the induced 2-tuple linguistic generalized ordered weighted averaging (2-TILGOWA) operator and generalized the 2-TILGOWA by using quasi-arithmetic means and Choquet integrals. On this basis, Halouani etc [27] defined 2-tuple choquet integral harmonic averaging (TCIHA), 2-tuple ordered choquet integral harmonic averaging (TOCIHA) and applied them to group decision making (GDM). Ju etc [28,29] proposed Trapezoid 2-tuple linguistic aggregation operator and new Shapley 2-tuple linguistic Choquet aggregation operators and applied to MADM; (2) 2-tuple linguistic aggregation operator related to Harmonic operators. Such as, Park etc [30] introduced linguistic harmonic (2TLH) operator, 2-tuple linguistic weighted harmonic (2TL-WH) operator, 2-tuple linguistic ordered weighted harmonic (2TLOWH) operator and 2-tuple linguistic hybrid harmonic (2TLHH) operator along with their properties. Wei proposed some Harmonic 2-tuple linguistic aggregation operator [31]; (3) Extended and induced 2-tuple linguistic aggregation operators. For example, Wan proposed 2-tuple linguistic hybrid arithmetic aggregation operators [32], Hybrid geometric aggregation operators [33] and applied them to MAGDM problems. Consequently, Meng and Tang [34] introduced the concepts of the extended 2-tuple linguistic hybrid arithmetical weighted (ET-LHAW) operator, the extended 2-tuple linguistic hybrid geometric mean (ET-LHGM) operator, the induced ET-LHAW (IET-LHAW) operator and the induced ET-LHGM (IET-LHGM) operator. Li etc introduced the induced aggregation operators and distance measures under the 2-tuple linguistic environment and built MADM method in [35]. Wei established a new MAGDM method based on extended 2-tuple linguistic weighted geometric aggregation (ET-WG) operator [36], extended 2-tuple linguistic ordered weighted geometric aggregation (ET-OWG) operator [36], some dependent 2-tuple linguistic aggregation operators [37]; (4) 2-tuple linguistic power aggregation operators. For example, Xu etc [38] studied the MAGDM method based on 2-tuple linguistic power aggregation operators (2TLPA) under linguistic environment, on basis of 2TLPA, Wu etc. [39] proposed some 2-tuple linguistic generalized power aggregation operators (2TLGPA); (5) Others 2-tuple linguistic aggregation operators. For instance, Xu etc. [40] established linguistic decision making methods based on proportional 2-tuple geometric weighted aggregation operators (PTWGA).

In order to develop an approach for consensus problems when expert preference information is in the form of uncertain linguistic preference relations, Xu etc [41] introduced the concept of uncertain 2-tuple linguistic variables and uncertain 2-tuple linguistic weighed averaging (ULWA$_{2-tuple}$) operator, and then Zhang [42] introduced uncertain 2-tuple linguistic preference relation. As far as the interval-valued 2-tuple linguistic aggregation operators are concerned, some new uncertain (or interval-valued) 2-tuple linguistic aggregation operators were proposed in many literatures. for instance, interval-valued 2-tuple aggregation operators [43], dependent interval 2-tuple linguistic aggregation operators [44], generalized interval-valued 2-tuple linguistic corrected aggregation operators [45,46], interval-valued 2-tuple power aggregation operators [47], interval 2-tuple linguistic Harmonic mean operators [48], interval 2-tuple linguistic Choquet integral aggregation operators [49] and some interval-valued 2-tuple linguistic aggregation operators [50,51]. Whilst, some kinds of MAGDM methods based on these aggregation operators were also developed.

Muirhead mean (MM) [52] is a well-known aggregation operator for it can consider the interrelationships among any number of aggregation arguments and it is also a universal operator since it contain other general operators by assessing different parameter vectors. In addition, MM is also a generalization of Maclaurin symmetric mean (MSM) [53]. When the parameter vector is assessed different values in MM, which will reduce to some existing operators. Many extensions of MM and MSM have been developed, e. g., intuitionistic fuzzy MM operators [54], 2-tuple linguistic MM operators [55], hesitant fuzzy Maclaurin symmetric mean [5,15]. Although MSM is a special situation of MM, in order to solve the more complicated decision making problems in real world, it is necessary and significant to develop uncertain 2-tuple linguistic MM (UTL-MM) based on MM that not only accommodate uncertain 2-tuple linguistic information but also can capture the interrelationships among multi-input arguments. Furthermore, UTL-MM can be considered a uniform form for some existing aggregation operators such as interval-valued 2-tuple weighted averaging (IVTWA)[50], interval-valued 2-tuple weighted geometric [51], uncertain linguistic weighted average (ULWA) operator [40] and so on.

The goal of this paper is to develop new method for MAGDM problems with uncertain 2-tuple linguistic information based on UTL-MM by combining MM and uncertain 2-tuple linguistic information. To do so, the rest of the paper is organized as follows. In Section 2, we review some definitions on linguistic term, 2-tuple linguistic variable, which are used in the analysis throughout this paper. Section 3 is devoted to the new uncertain 2-tuple linguistic representation model. Section 4 is focused on uncertain 2-tuple linguistic weighted Muirhead mean (UTL-WMM) Operator along with their properties and some special cases. In Section 5, extend UTL-WMM operator to extended uncertain 2-tuple linguistic weighted Muirhead mean (EUTL-WMM) operators in order to deal with some decision making problems with uncertain 2-tuple linguistic information whose attribute values are expressed in uncertain 2-tuple linguistic information and the attribute weight is also 2-tuple linguistic information. In Section 6, we construct a MAGDM approach based on UTL-WMM and EUTL-WMM operators proposed in Section 4 and Section 5. Consequently, a practical example is provided in Section 7 to verify the validity of the proposed method and to show their advantages. In Section 8, we give some conclusions of this study.

2. Uncertain 2-tuple Linguistic Representation Model

In this section, some fundamental concepts of (uncertain) 2-tuple linguistic models are recapped, they are the basis of this work.

Let $S = \{s_i | i = 0, 1, \cdots, g\}$ be a linguistic term set with odd cardinality, for any label s_i, which represents a possible values for a linguistic variable and satisfy the following characteristics [1]:

(1)$s_i > s_j$ if and only if $i > j$;

(2) if $s_i \geqslant s_j$, then $max(s_i, s_j) = s_i$;

(3) if $s_i \geqslant s_j$, then $min(s_i, s_j) = s_j$;

(4) $Neg(s_i) = s_j$, such that $j = g - i$.

To compute with words without loss of information, the 2-tuple linguistic model based on the concept of symbolic translation was proposed in [1,2,56,57]. The model uses a 2-tuple (s_k, α) to represent linguistic information, where $s_k \in S$, α denotes the value of symbolic translation and $\alpha \in [-0.5, 0.5)$. The specific definition of 2-tuple linguistic model is given as follows.

Definition 1. [1] Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set and $\beta \in [0, g]$ be a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to β is obtained with the following function:

$$\Delta : [0, g] \to S \times [-0.5, 0.5)$$
$$\Delta(\beta) = (s_i, \alpha), with \begin{cases} s_i, & i = round(\beta) \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5) \end{cases}$$

where $round()$ is the usual round operation, s_i has the closest index label to β and α is the value of symbolic translation.

Definition 2. [1] Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set and (s_i, α) be a 2-tuple, there is a function Δ^{-1}, which can transform a 2-tuple into its equivalent numerical value $\beta \in [0, g]$. The transformation function can be defined as

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [0, g]$$
$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta.$$

It easily follows from Def. 1 and Def. 2 that a linguistic term can be considered as a linguistic 2-tuple by adding a value 0 to it as symbolic translation, i.e. $\Delta(s_i) = (s_i, 0)$.

For example, let $S = \{s_0 =$ extremely poor (EP), $s_1 =$ very poor (VP), $s_2 =$ poor (P), $s_3 =$ slightly poor (SP), $s_4 =$ Medium (M), $s_5 =$ slightly good (S-G), $s_6 =$ good (G), $s_7 =$ very good (VG), $s_8 =$ extremely good (EG) $\}$ be a linguistic term set. If a decision-maker thinks that the profit of a project is`very good´, then we can change this assessment into a 2-tuple $(s_7, 0)$. However, if a decision-maker thinks that the profit of a project is`at most medium´, then above linguistic model will fail to deal with this situation. In order to solve this limitation, Xu [41] introduced uncertain 2-tuple linguistic variable which is defined as follows:

Definition 3. [41] Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set with odd cardinality, then an uncertain linguistic variable can be denoted by $[s^-, s^+]$, where $s^-, s^+ \in S$, s^- and s^+ are the lower and upper limits of the uncertain linguistic variable. In particular, if $s^- = s^+$, then $[s^-, s^+]$ will reduce to a linguistic term s^-.

For example, in the above example, we can use the uncertain linguistic variable $[s_0, s_4]$ to express this evaluation`at most medium´. When $s^- = s^+$, the uncertain linguistic variable will reduce to linguistic variable. Therefore, uncertain linguistic variable is a kind of useful extension of linguistic variable. Based on the 2-tuple linguistic model, Zhang [42] defined the uncertain 2-tuple linguistic variable:

Definition 4. [42] Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set with odd cardinality, then an uncertain 2-tuple linguistic variable can be denoted by $[(s^-, \alpha^-), (s^+, \alpha^+)]$, $s^- \leqslant s^+$, where $(s^-, \alpha^-), (s^+, \alpha^+) \in S \times [-0.5, 0.5)$, (s^-, α^-) and (s^+, α^+) are the lower and upper limits of the uncertain 2-tuple linguistic variable.

3. Modified Uncertain 2-Tuple Linguistic Representation Model

Although the uncertain 2-tuple linguistic variable was introduced, its representation model is not given. In this section, after analyzing the generalized 2-tuple model, we introduce the uncertain 2-tuple linguistic representation model based on uncertain 2-tuple linguistic variable and give a comparison rule of two uncertain 2-tuples on linguistic terms set with multi-granularity.

To deal with linguistic information from different linguistic term sets, Chen and Tai [59] proposed a generalized 2-tuple linguistic model and translation functions.

Definition 5. [59] Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set, then the 2-tuple can be obtained by the translation function θ:

$$\theta : S \rightarrow S \times [-0.5/g, 0.5/g),$$
$$\theta(s_i) = (s_i, 0), for\ \ any\ \ s_i \in S.$$

Definition 6. [59] Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set and $\beta \in [0, g]$ be a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to β is obtained with the following function:

$$\Delta : [0,1] \rightarrow S \times [-0.5/g, 0.5/g)$$
$$\Delta(\beta) = (s_i, \alpha),$$

with

$$\begin{cases} s_i, & i = round(\beta g) \\ \alpha = \beta - i/g, & \alpha \in [-0.5/g, 0.5/g) \end{cases}$$

where $round()$ is the usual round operation, s_i has the closest index label to β and α is the value of symbolic translation.

Definition 7. [59] Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set and (s_i, α) be a 2-tuple, there is a function Δ^{-1}, which can transform a 2-tuple into its equivalent numerical value $\beta \in [0, 1]$. The transformation function can be defined as

$$\Delta^{-1} : S \times [-0.5/g, 0.5/g) \rightarrow [0,1]$$
$$\Delta^{-1}(s_i, \alpha) = i/g + \alpha = \beta.$$

We can see from Definitions 7 that $\beta \in [0, 1]$, the main advantage of this assignment method of β is that it is very convenient to compare and aggregate 2-tuples from different linguistic term sets. Herrera etc. [2] proposed 2-tuple linguistic model, in which the linguistic term s_i in a 2-tuple (s_i, α) has the closest index label to the symbolic aggregation value β, and symbolic translation $\alpha \in [-0.5, 0.5)$ represents the deviation value of i and β. So the aggregation result represented by a 2-tuple has a clear implication. However, Wei [58] pointed out that the symbolic translation α in a 2-tuple (s_i, α) defined in Def. 5, Def. 6 and Def. 7 is in $[-0.5/g, 0.5/g)$ and the meaning of its expression is not clear. Thus, in order to deal with the linguistic information and describe the aggregation result, Wei [58] modified the translation functions as follows:

Definition 8. [58] Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set and $\beta \in [0, g]$ be a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to β is obtained with the following function:

$$\Delta : [0,1] \rightarrow S \times [-0.5, 0.5)$$
$$\Delta(\beta) = (s_i, \alpha),$$

with

$$\begin{cases} s_i, & i = round(\beta g), \\ \alpha = \beta g - i, & \alpha \in [-0.5, 0.5). \end{cases}$$

where $round()$ is the usual round operation, s_i has the closest index label to β and α is the value of symbolic translation.

Definition 9. Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set and (s_i, α) be a 2-tuple, there is a function Δ^{-1}, which can transform a 2-tuple into its equivalent numerical value $\beta \in [0, 1]$. The transformation function can be defined as

$$\Delta^{-1} : S \times [-0.5/g, 0.5/g) \rightarrow [0,1]$$
$$\Delta^{-1}(s_i, \alpha) = (i + \alpha)/g = \beta.$$

Although the uncertain 2-tuple linguistic variable was proposed by Xu [41], whose representation model do not been given. Motivated by interval-valued 2-tuple linguistic representation model [50], we put forward the uncertain 2-tuple linguistic representation model based on Def. 8 and Def. 9.

Definition 10. Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set. An interval-valued 2-tuple is composed of two linguistic terms and two crisp numbers, denoted by $(s_i, \alpha_1), (s_j, \alpha_2)$, where $i \leqslant j$ and $\alpha_1 \leqslant \alpha_2$ if $i = j$. s_i, s_j represent the linguistic label of the linguistic term set S and α_1, α_2 represent the symbol translation. The uncertain 2-tuple that express the equivalent information to an interval value

$[\beta_1, \beta_2](\beta_1, \beta_2 \in [0,1], \beta_1 \leqslant \beta_2)$ is derived by the following function

$$\Delta([\beta_1, \beta_2]) = [(s_i, \alpha_1), (s_j, \alpha_2)],$$

with

$$\begin{cases} s_i, & i = round(\beta_1 g), \\ s_j, & j = round(\beta_2 g), \\ \alpha_1 = \beta_1 g - i, & \alpha_1 \in [-0.5, 0.5), \\ \alpha_2 = \beta_2 g - j, & \alpha_2 \in [-0.5, 0.5). \end{cases} \tag{1}$$

Conversely, there exist a function Δ^{-1} such that uncertain 2-tuple can be translated into an interval $[\beta_1, \beta_2](\beta_1, \beta_2 \in [0,1], \beta_1 \leqslant \beta_2)$ as follows:

$$\begin{aligned} \Delta^{-1}[(s_i, \alpha_1), (s_j, \alpha_2)] \quad &= [(\alpha_1 + i)/g, (\alpha_2 + j)/g] \\ &= [\beta_1, \beta_2]. \end{aligned} \tag{2}$$

If $s_i = s_j$ and $\alpha_1 = a_2$, Def. 10 will reduce to Def. 8 and Def. 9. In the following sections, the translation functions Δ and Δ^{-1} defined by Eq.(1) and Eq.(2) can help us to aggregate the multigranularity linguistic information. Since the function Δ^{-1} translates uncertain 2-tuples on different linguistic term sets into their normalized aggregation values, their comparisons can be carried out according to the following rules:

Let $S = \{s_0, s_1, \cdots, s_\tau\}$ be a linguistic term set with granularity $g = \tau + 1$. For an uncertain 2-tuple $A = [(s_i, \alpha_1), (s_j, \alpha_2)]$ on the linguistic term set S, the score function of A is defined as follows:

$$S^g(A) = \frac{1}{2}(\Delta^{-1}(s_i, \alpha_1) + \Delta^{-1}(s_j, \alpha_2)). \tag{3}$$

The accuracy function of A is defined as follows:

$$H^g(A) = \Delta^{-1}(s_j, \alpha_2) - \Delta^{-1}(s_i, \alpha_1)). \tag{4}$$

It is obvious that $S(A) \in [0,1]$ and $H(A) \in [0,1]$. Now, the compare rule of two uncertain 2-tuple is listed as follows:

Let S_{g_1} and S_{g_2} be two linguistic term sets with granularity g_1 and g_2, respectively. And A, B are two uncertain 2-tuples on S_{g_1}, S_{g_2}, respectively.

If $S^{g_1}(A) > S^{g_2}(B)$, then $A > B$;
If $S^{g_1}(A) < S^{g_2}(B)$, then $A < B$;
If $S^{g_1}(A) = S^{g_2}(B)$, then:
(1) $H^{g_1}(A) > H^{g_2}(B)$, then $A > B$;
(2) $H^{g_1}(A) < H^{g_2}(B)$, then $A < B$;
(3) $H^{g_1}(A) = H^{g_2}(B)$, then $A = B$.

Example 1. let $A = [(s_4, 0.1), (s_5, 0.2)]$ and $B = [(s_3, 0.2), (s_4, -0.1)]$ be two 2-tuples on linguistic term sets S^7 and S^5, respectively. Since

$$\begin{aligned} S^7(A) &= \frac{1}{2}(\Delta^{-1}(s_4, 0.1) + \Delta^{-1}(s_5, 0.2)) = 0.7417; \\ S^5(B) &= \frac{1}{2}(\Delta^{-1}(s_3, 0.2) + \Delta^{-1}(s_4, -0.1)) = 0.8875, \end{aligned}$$

we have $B > A$.

4. Uncertain 2-Tuple Linguistic Weighted Muirhead Mean Operator

In this section, firstly, we recall the traditional Muirhead mean (MM) operator which can only process the crisp number. And 2-tuple linguistic model can avoid the information loss in the process of linguistic information processing, so it is necessary to extend traditional MM to uncertain linguistic environment in order to deal with some decision making problems with uncertain 2-tuple linguistic information. In this section, we will propose some uncertain 2-tuple linguistic Muirhead mean operators and uncertain 2-tuple linguistic weighted Muirhead mean operators for the uncertain 2-tuple linguistic information, investigate some properties of the new operators and obtain some special cases of uncertain 2-tuple linguistic MM operator when the parameter vector takes different values.

4.1. Muirhead Mean Operator

The Muirhead mean (MM) operator [52] is a general aggregation function and firstly proposed by Muirhead in 1902, it is defined as follows:

Definition 11.[52] Let $a_i (i = 1, 2, \cdots, n)$ be a collection of nonnegative real numbers, $A = \{a_1, a_2, \cdots, a_n\}$ and $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector, if

$$MM^P(a_1, \cdots, a_n) = (\frac{1}{n!}(\sum_{\theta \in S_n}(\prod_{j=1}^{n} a_{\theta(j)}^{p_j})))^{\frac{1}{\Sigma_{j=1}^{n} p_j}}, \tag{5}$$

The we call MM^P the Muirhead mean (MM), where $\theta(j)(j=1,2,\cdots,n)$ is any permutation of $(1,2,\cdots,n)$ and S_n is the collection of all permutation of $\theta(j)(j=1,2,\cdots,n)$.

There are some special cases when the parameter vector assessed different values.

(1) If $P=(1,0,\cdots,0)$, MM operator will reduce to arithmetic averaging operator

$$MM^{(1,0,\cdots,0)}(a_1,\cdots,a_n)=\frac{1}{n}\sum_{j=1}^{n}a_j. \tag{6}$$

(2) If $P=(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})$, PFLMM operator will reduce to Maclaurin symmetric mean (MSM) operator

$$PFLMM^{(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})}(a_1,\cdots,a_n)$$
$$=(\frac{\sum_{1\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant n}\prod_{j=1}^{k}a_j}{C_n^k})^{\frac{1}{k}}; \tag{7}$$

(3) If $P=(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$, MM operator will reduce to geometric averaging operator

$$MM^{(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})}(a_1,\cdots,a_n)=\prod_{j=1}^{n}a_j^{\frac{1}{n}}. \tag{8}$$

We can see from the above discussion that and MM operator is a generalization of most existing aggregation operators, the main advantage of the MM operator is that it can capture the interrelationships among the multiple aggregated arguments. Now, we extend the traditional MM operator to uncertain 2-tuple linguistic environment in order to solve more complex decision problems with uncertain linguistic information.

4.2. Uncertain 2-Tuple Linguistic Muirhead Mean Operator

Definition 12. Let $\{\widetilde{b}_i=[(r_i,\alpha_i),(l_i,\beta_i)]|i=1,2,\cdots,n\}$ be the set of n uncertain 2-tuple linguistic variables and $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector. Then the uncertain 2-tuple linguistic Muirhead mean operator (UTL-MM)is defined as follows:

$$UTL-MM^P(\widetilde{b}_1,\widetilde{b}_2,\cdots,\widetilde{b}_n)=UTL-MM^P([(r_1,\alpha_1),(l_1,\beta_1)],\cdots,[(r_n,\alpha_n),(l_n,\beta_n)])$$
$$=\Delta[(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(r_{\theta(j)},\alpha_{\theta(j)}))^{p_j})))^{\frac{1}{\sum_{j=1}^{n}p_j}},(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(l_{\theta(j)},\beta_{\theta(j)}))^{p_j})))^{\frac{1}{\sum_{j=1}^{n}p_j}}], \tag{9}$$

where $\theta(j)(j=1,2,\cdots,n)$ is any permutation of $(1,2,\cdots,n)$ and S_n is the collection of all permutations of $\theta(j)(j=1,2,\cdots,n)$.

Example 2. Let $S=\{s_0,s_1,\cdots,s_6\}$ be a linguistic term set and $\{\widetilde{b}_1=[(s_1,-0.2),(s_2,0.1)],\widetilde{b}_2=[(s_3,0.1),(s_4,0.3)],\widetilde{b}_3=[(s_5,-0.3),(s_6,-0.1)]\}$ be set of three uncertain 2-tuple linguistic variables and $P=(\frac{1}{2},\frac{1}{3},\frac{1}{6})$. Let

$$UTL-MM^P(\widetilde{b}_1,\widetilde{b}_2,\widetilde{b}_3)=\Delta[a,b].$$

According to Eq. (9), we have

$$\begin{aligned} a &= (\frac{1}{3!}(0.13^{\frac{1}{2}} \times 0.52^{\frac{1}{3}} \times 0.78^{\frac{1}{6}} + 0.13^{\frac{1}{2}} \times 0.78^{\frac{1}{3}} \times 0.51^{\frac{1}{6}} + 0.52^{\frac{1}{2}} \times 0.13^{\frac{1}{3}} \times 0.78^{\frac{1}{6}} \\ &\quad + 0.52^{\frac{1}{2}} \times 0.78^{\frac{1}{3}} \times 0.13^{\frac{1}{6}} + 0.78^{\frac{1}{2}} \times 0.13^{\frac{1}{3}} \times 0.52^{\frac{1}{6}} + 0.78^{\frac{1}{2}} \times 0.52^{\frac{1}{3}} \times 0.13^{\frac{1}{6}}))^{\frac{1}{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}} \\ &= 0.3869. \\ b &= (\frac{1}{3!}(0.35^{\frac{1}{2}} \times 0.72^{\frac{1}{3}} \times 0.98^{\frac{1}{6}} + 0.35^{\frac{1}{2}} \times 0.98^{\frac{1}{3}} \times 0.72^{\frac{1}{6}} + 0.72^{\frac{1}{2}} \times 0.35^{\frac{1}{3}} \times 0.98^{\frac{1}{6}} \\ &\quad + 0.72^{\frac{1}{2}} \times 0.98^{\frac{1}{3}} \times 0.35^{\frac{1}{6}} + 0.98^{\frac{1}{2}} \times 0.35^{\frac{1}{3}} \times 0.72^{\frac{1}{6}} + 0.98^{\frac{1}{2}} \times 0.72^{\frac{1}{3}} \times 0.35^{\frac{1}{6}}))^{\frac{1}{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}} \\ &= 0.6320. \end{aligned}$$

Therefore,

$$UTL-MM^P(\widetilde{b}_1,\widetilde{b}_2,\widetilde{b}_3) = \Delta[0.3869, 0.6320] \\ = [(s_2, 0.3216),(s_4,-0.2078)].$$

In the process of decision making, the aggregation results would be more reliable if the selected operator is monotonic, the lack of monotonicity may debase the reliability and dependability of the final decision-making results. Next, we can prove the $UTL-MM^P(\widetilde{b}_1,\widetilde{b}_2,\cdots,\widetilde{b}_n)$ is idempotent, bounded, and monotonic.

Theorem 1. Let $\{\widetilde{b}_i = [(r_i,\alpha_i),(l_i,\beta_i)]|i = 1,2,\cdots,n\}$ be the set of n uncertain 2-tuple linguistic variables and $P = (p_1, p_2,\cdots, p_n) \in \mathbf{R}^n$ be a parameter vector. If $\widetilde{b}_i = [(r_i,\alpha_i),(l_i,\beta_i)] = \widetilde{b} = [(r,\alpha),(l,\beta)](i = 1,2,\cdots,n)$, then

$$UTL-MM^P(\widetilde{b}_1,\widetilde{b}_2,\cdots,\widetilde{b}_n) = \widetilde{b}.$$

Proof. Since $\widetilde{b}_i = [(r_i,\alpha_i),(l_i,\beta_i)] = \widetilde{b} = [(r,\alpha),(l,\beta)](i = 1,2,\cdots,n)$, we have

$$\begin{aligned} &UTL-MM^P(\widetilde{b}_1,\widetilde{b}_2,\cdots,\widetilde{b}_n) = UTL-MM^P([(r_1,\alpha_1),(l_1,\beta_1)],\cdots,[(r_n,\alpha_n),(l_n,\beta_n)]) \\ &= \Delta[(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(r,\alpha))^{p_j})))^{\frac{1}{\sum_{j=1}^n p_j}}, (\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(l,\beta))^{p_j})))^{\frac{1}{\sum_{j=1}^n p_j}}] \\ &= \Delta[(\frac{1}{n!}(\sum_{\theta\in S_n}((\Delta^{-1}(r,\alpha))^{\sum_{j=1}^n p_j})))^{\frac{1}{\sum_{j=1}^n p_j}}, (\frac{1}{n!}(\sum_{\theta\in S_n}((\Delta^{-1}(l,\beta))^{\sum_{j=1}^n p_j})))^{\frac{1}{\sum_{j=1}^n p_j}}] \\ &= \Delta[(\frac{1}{n!}(n!((\Delta^{-1}(r,\alpha))^{\sum_{j=1}^n p_j})))^{\frac{1}{\sum_{j=1}^n p_j}}, (\frac{1}{n!}(n!((\Delta^{-1}(l,\beta))^{\sum_{j=1}^n p_j})))^{\frac{1}{\sum_{j=1}^n p_j}}] \\ &= \Delta[((\Delta^{-1}(r,\alpha))^{\sum_{j=1}^n p_j})^{\frac{1}{\sum_{j=1}^n p_j}}, ((\Delta^{-1}(l,\beta))^{\sum_{j=1}^n p_j})^{\frac{1}{\sum_{j=1}^n p_j}}] \\ &= \Delta[(\Delta^{-1}(r,\alpha), \Delta^{-1}(l,\beta)] = \widetilde{b}. \end{aligned}$$

Theorem 2.(Monotonicity) Let $\{\widetilde{b}_i = [(r_i,\alpha_i),(l_i,\beta_i)]|i = 1,2,\cdots,n\}$, $\{\widetilde{b}_i' = [(r_i',\alpha_i'),(l_i',\beta_i')]|i = 1,2,\cdots,n\}$ be the two sets of n uncertain 2-tuple linguistic variables and $P = (p_1, p_2,\cdots, p_n) \in \mathbf{R}^n$ be a parameter vector. If $(r_i,\alpha_i) \geqslant (r_i',\alpha_i')$ and $(l_i,\beta_i) \geqslant (l_i',\beta_i')$ for any $i(i = 1,2,\cdots,n)$, then

$$UTL-MM^P(b_1,b_2,\cdots,b_n) \\ \geqslant UTL-MM^P(b_1',b_2',\cdots,b_n').$$

Proof. Since $(r_i,\alpha_i) \geqslant (r_i',\alpha_i')$ and $(l_i,\beta_i) \geqslant$

$(l_i^{'},\beta_i^{'})$, we have

$$\Delta^{-1}(r_{\theta(j)},\alpha_{\theta(j)}) \geqslant \Delta^{-1}(r_{\theta(j)}^{'},\alpha_{\theta(j)}^{'}),\Delta^{-1}(l_{\theta(j)},\beta_{\theta(j)}) \geqslant \Delta^{-1}(l_{\theta(j)}^{'},\beta_{\theta(j)}^{'}).$$

and so

$$(\Delta^{-1}(r_{\theta(j)},\alpha_{\theta(j)}))^{p_j} \geqslant (\Delta^{-1}(r_{\theta(j)}^{'},\alpha_{\theta(j)}^{'}))^{p_j},$$
$$(\Delta^{-1}(l_{\theta(j)},\beta_{\theta(j)}))^{p_j} \geqslant (\Delta^{-1}(l_{\theta(j)}^{'},\beta_{\theta(j)}^{'}))^{p_j}.$$

and

$$\prod_{j=1}^{n}(\Delta^{-1}(r_{\theta(j)},\alpha_{\theta(j)}))^{p_j} \geqslant \prod_{j=1}^{n}(\Delta^{-1}(r_{\theta(j)}^{'},\alpha_{\theta(j)}^{'}))^{p_j},$$
$$\prod_{j=1}^{n}(\Delta^{-1}(l_{\theta(j)},\beta_{\theta(j)}))^{p_j} \geqslant \prod_{j=1}^{n}(\Delta^{-1}(l_{\theta(j)}^{'},\beta_{\theta(j)}^{'}))^{p_j}.$$

So, we obtain

$$\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(r_{\theta(j)},\alpha_{\theta(j)}))^{p_j}) \geqslant \sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(r_{\theta(j)}^{'},\alpha_{\theta(j)}^{'}))^{p_j}),$$
$$\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(l_{\theta(j)},\beta_{\theta(j)}))^{p_j}) \geqslant \sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(l_{\theta(j)}^{'},\beta_{\theta(j)}^{'}))^{p_j}).$$

And so

$$(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(r_{\theta(j)},\alpha_{\theta(j)}))^{p_j})))^{\frac{1}{\Sigma_{j=1}^{n}p_j}} \geqslant (\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(r_{\theta(j)}^{'},\alpha_{\theta(j)}^{'}))^{p_j})))^{\frac{1}{\Sigma_{j=1}^{n}p_j}}$$
$$(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(l_{\theta(j)},\beta_{\theta(j)}))^{p_j})))^{\frac{1}{\Sigma_{j=1}^{n}p_j}} \geqslant (\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(l_{\theta(j)}^{'},\beta_{\theta(j)}^{'}))^{p_j})))^{\frac{1}{\Sigma_{j=1}^{n}p_j}},$$

that is,

$$UTL-MM^{P}(b_1,b_2,\cdots,b_n) \geqslant UTL-MM^{P}(b_1^{'},b_2^{'},\cdots,b_n^{'}).$$

Theorem 3. **(Boundeness)** Let $\{\widetilde{b}_i = [(r_i,\alpha_i),(l_i,\beta_i)]|i = 1,2,\cdots,n\}$ be the set of n uncertain 2-tuple linguistic variables and $P = (p_1,p_2,\cdots,p_n) \in \mathbf{R}^n$ be a parameter vector. then

$$\Delta[min_i(r_i,\alpha_i),min_i(l_i,\beta_i)] \leqslant h^{-} \leqslant UTL-MM^{P}(\widetilde{b}_1,\cdots,\widetilde{b}_n) \leqslant \Delta[max_i(r_i,\alpha_i),max_i(l_i,\beta_i)].$$

Proof. Since $min_i(r_i,\alpha_i) \leqslant max_i(r_i,\alpha_i)$ and $min_i(l_i,\beta_i) \leqslant max_i(l_i,\beta_i)$, it is easy to prove the Boundness of UTL-MM operator according to Theorem 1 and Theorem 2.

From the Definition of UTL-MM operator, it is easy to verify the commutativity of the operator, that is:

Theorem 4. **(Commutativity)** Let $\{\widetilde{b}_i = [(r_i,\alpha_i),(l_i,\beta_i)]|i = 1,2,\cdots,n\}$, $\{\widetilde{b}_i^{'} = [(r_i^{'},\alpha_i^{'}),(l_i^{'},\beta_i^{'})]|i = 1,2,\cdots,n\}$ be the two sets of n uncertain 2-tuple linguistic variables and $P = (p_1,p_2,\cdots,p_n) \in \mathbf{R}^n$ be a parameter vector. If $\{\widetilde{b}_i^{'} = [(r_i^{'},\alpha_i^{'}),(l_i^{'},\beta_i^{'})]|i = 1,2,\cdots,n\}$ is any permutation of $\{\widetilde{b}_i = [(r_i,\alpha_i),(l_i,\beta_i)]|i = 1,2,\cdots,n\}$, then

$$UTL-MM^{P}(\widetilde{b}_1,\cdots,\widetilde{b}_n) = UTL-MM^{P}(\widetilde{b}_1^{'},\cdots,\widetilde{b}_n^{'}).$$

Now, we develop some special cases of UTL-MM operator with respect to the different parameter vector P. Let $\{\widetilde{b}_i = [(r_i,\alpha_i),(l_i,\beta_i)]|i = 1,2,\cdots,n\}$ and $P = (p_1,p_2,\cdots,p_n) \in \mathbf{R}^n$ be a parameter vector.

(1) If $P = (1,0,\cdots,0)$, UTL-MM operator will reduce to uncertain 2-tuple linguistic average (UTLA) operator [50]

$$UTL-MM^{(1,0,\cdots,0)}(\widetilde{b}_1,\cdots,\widetilde{b}_n) = \Delta[\frac{1}{n}\sum_{j=1}^{n}\Delta^{-1}(r_j,\alpha_j),\frac{1}{n}\sum_{j=1}^{n}\Delta^{-1}(l_j,\beta_j)] \quad (10)$$

(2) If $P = (\lambda,0,\cdots,0)$, UTL-MM operator will reduce to generalized uncertain 2-tuple linguistic average (GUTLA) operator [51]

$$UTL-MM^{(1,0,\cdots,0)}(\widetilde{b}_1,\cdots,\widetilde{b}_n) = \Delta[(\frac{1}{n}\sum_{j=1}^{n}(\Delta^{-1}(r_j,\alpha_j))^{\lambda})^{\frac{1}{\lambda}},(\frac{1}{n}\sum_{j=1}^{n}(\Delta^{-1}(l_j,\beta_j))^{\lambda})^{\frac{1}{\lambda}}]. \quad (11)$$

(3) If $P = (\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})$, UTL-MM operator will reduce to uncertain 2-tuple linguistic Maclaurin symmetric mean (UTL-MSM) operator

$$
\begin{aligned}
& UTL-MSM^{(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})}(\widetilde{b}_1,\cdots,\widetilde{b}_n) \\
&= \Delta[(\frac{k!(n-k)!}{n!}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}(\prod_{j=1}^{n}\Delta^{-1}(r_{i_j},\alpha_{i_j})))^{\frac{1}{k}},(\frac{k!(n-k)!}{n!}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}(\prod_{j=1}^{n}\Delta^{-1}(l_{i_j},\beta_{i_j})))^{\frac{1}{k}}] \\
&= \Delta[(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}(\prod_{j=1}^{n}\Delta^{-1}(r_{i_j},\alpha_{i_j})))^{\frac{1}{k}},(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}(\prod_{j=1}^{n}\Delta^{-1}(l_{i_j},\beta_{i_j})))^{\frac{1}{k}}]. \qquad (12)
\end{aligned}
$$

(4) If $P=(1,1,\cdots,1)$, UTL-MM operator will reduce to uncertain 2-tuple linguistic geometric (UTLG) operator [51]

$$
\begin{aligned}
& UTL-MM^{(1,1,\cdots,1)}(\widetilde{b}_1,\cdots,\widetilde{b}_n) \\
&= \Delta[(\prod_{j=1}^{n}\Delta^{-1}(r_j,\alpha_j))^{\frac{1}{n}},(\prod_{j=1}^{n}\Delta^{-1}(l_j,\beta_j))^{\frac{1}{n}}] \qquad (13)
\end{aligned}
$$

(5) If $P=(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$, UTL-MM operator will reduce to uncertain 2-tuple linguistic geometric (UTLG) operator [51]

$$
\begin{aligned}
& UTL-MM^{(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})}(\widetilde{b}_1,\cdots,\widetilde{b}_n) \\
&= \Delta[(\prod_{j=1}^{n}\Delta^{-1}(r_j,\alpha_j))^{\frac{1}{n}},(\prod_{j=1}^{n}\Delta^{-1}(l_j,\beta_j))^{\frac{1}{n}}] \qquad (14)
\end{aligned}
$$

4.3. Uncertain 2-tuple Linguistic Weighted Muirhead Mean Operators

Weights of attributes play a vital role in decision making and will directly the results of decision making results. In this Section, we propose the UTL-MM aggregation operators which can not consider the weights of attributes, so it is very important to consider to weights of attributes in the process of information aggregation.

Definition 13. Let $\{\widetilde{b}_i=[(r_i,\alpha_i),(l_i,\beta_i)]|i=1,2,\cdots,n\}$ be the set of n uncertain 2-tuple linguistic variables and $(w_1,\cdots,w_n)^T$ be their associated weights with $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector. Then the uncertain 2-tuple linguistic weighted Muirhead mean operator (UTL-WMM)is defined as follows:

$$
\begin{aligned}
& UTL-WMM^{P}(\widetilde{b}_1,\widetilde{b}_2,\cdots,\widetilde{b}_n)=UTL-WMM^{P}([(r_1,\alpha_1),(l_1,\beta_1)],\cdots,[(r_n,\alpha_n),(l_n,\beta_n)]) \\
&= \Delta[(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(nw_{\theta(j)}\Delta^{-1}(r_{\theta(j)},\alpha_{\theta(j)}))^{p_j})))^{\frac{1}{\sum_{j=1}^{n}p_j}},(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(nw_{\theta(j)}\Delta^{-1}(l_{\theta(j)},\beta_{\theta(j)}))^{p_j})))^{\frac{1}{\sum_{j=1}^{n}p_j}}], \qquad (15)
\end{aligned}
$$

where $\theta(j)(j=1,2,\cdots,n)$ is any permutation of $(1,2,\cdots,n)$ and S_n is the collection of all permutations of $\theta(j)(j=1,2,\cdots,n)$.

Example 3. Let $S=\{s_0,s_1,\cdots,s_6\}$ be a linguistic term set and $\{\widetilde{b}_1=[(s_1,-0.2),(s_2,0.1)],\widetilde{b}_2=[(s_3,0.1),(s_4,0.3)],\widetilde{b}_3=[(s_5,-0.3),(s_6,-0.1)]\}$ be set of three uncertain 2-tuple linguistic variables with weights vector (0.4, 0.3, 0.3) and $P=(\frac{1}{2},\frac{1}{3},\frac{1}{6})$. Let

$$UTL-MM^{P}(\widetilde{b}_1,\widetilde{b}_2,\widetilde{b}_3)=\Delta[a,b].$$

According to Eq. (15), we have

$$
\begin{aligned}
a &= (\frac{1}{3!}(0.16^{\frac{1}{2}}\times 0.47^{\frac{1}{3}}\times 0.71^{\frac{1}{6}}+0.16^{\frac{1}{2}}\times 0.71^{\frac{1}{3}}\times 0.47^{\frac{1}{6}}+0.47^{\frac{1}{2}}\times 0.16^{\frac{1}{3}}\times 0.71^{\frac{1}{6}}\\
&\quad +0.47^{\frac{1}{2}}\times 0.71^{\frac{1}{3}}\times 0.16^{\frac{1}{6}}+0.71^{\frac{1}{2}}\times 0.16^{\frac{1}{3}}\times 0.47^{\frac{1}{6}}+0.71^{\frac{1}{2}}\times 0.47^{\frac{1}{3}}\times 0.16^{\frac{1}{6}}))^{\frac{1}{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}}\\
&= 0.3804.\\
b &= (\frac{1}{3!}(0.42^{\frac{1}{2}}\times 0.65^{\frac{1}{3}}\times 0.89^{\frac{1}{6}}+0.42^{\frac{1}{2}}\times 0.89^{\frac{1}{3}}\times 0.65^{\frac{1}{6}}+0.65^{\frac{1}{2}}\times 0.42^{\frac{1}{3}}\times 0.89^{\frac{1}{6}}\\
&\quad +0.65^{\frac{1}{2}}\times 0.89^{\frac{1}{3}}\times 0.42^{\frac{1}{6}}+0.89^{\frac{1}{2}}\times 0.42^{\frac{1}{3}}\times 0.65^{\frac{1}{6}}+0.89^{\frac{1}{2}}\times 0.65^{\frac{1}{3}}\times 0.42^{\frac{1}{6}}))^{\frac{1}{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}}\\
&= 0.6236.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
UTL-MM^{P}(\widetilde{b}_1,\widetilde{b}_2,\widetilde{b}_3) &= \Delta[0.3804, 0.6236]\\
&= [(s_2, 0.2826), (s_4, -0.2581)].
\end{aligned}
$$

If the weight vector $w=(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$ in Def. 13, we have

$$
\begin{aligned}
&UTL-WMM^{P}([(r_1,\alpha_1),(l_1,\beta_1)],\cdots,[(r_n,\alpha_n),(l_n,\beta_n)])\\
&=\Delta[(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}((n\times\frac{1}{n})\Delta^{-1}(r_{\theta(j)},\alpha_{\theta(j)}))^{p_j})))^{\frac{1}{\Sigma_{j=1}^{n}p_j}},(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}((n\times\frac{1}{n})\Delta^{-1}(l_{\theta(j)},\beta_{\theta(j)}))^{p_j})))^{\frac{1}{\Sigma_{j=1}^{n}p_j}}],\\
&=\Delta[(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(r_{\theta(j)},\alpha_{\theta(j)}))^{p_j})))^{\frac{1}{\Sigma_{j=1}^{n}p_j}},(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\Delta^{-1}(l_{\theta(j)},\beta_{\theta(j)}))^{p_j})))^{\frac{1}{\Sigma_{j=1}^{n}p_j}}],\\
&=UTL-MM^{P}([(r_1,\alpha_1),(l_1,\beta_1)],\cdots,[(r_n,\alpha_n),(l_n,\beta_n)]).
\end{aligned}
$$

That is,

Theorem 5. UTL-MM operator is a special case of the UTL-WMM operator.

Similar to Theorem 2 and Theorem 3, we can prove $UTL-WMM^{P}(\widetilde{b}_1,\cdots,\widetilde{b}_n)$ are bounded, and monotonic.

Theorem 6. (Monotonicity) Let $\{\widetilde{b}_i=[(r_i,\alpha_i),(l_i,\beta_i)]|i=1,2,\cdots,n\}$, $\{\widetilde{b}_i^{'}=[(r_i^{'},\alpha_i^{'}),(l_i^{'},\beta_i^{'})]|i=1,2,\cdots,n\}$ be the two sets of n uncertain 2-tuple linguistic variables and $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector. If $(r_i,\alpha_i)\geqslant(r_i^{'},\alpha_i^{'})$ and $(l_i,\beta_i)\geqslant(l_i^{'},\beta_i^{'})$ for any $i(i=1,2,\cdots,n)$, then

$$
\begin{aligned}
&UTL-WMM^{P}(b_1,b_2,\cdots,b_n)\\
&\geqslant UTL-WMM^{P}(b_1^{'},b_2^{'},\cdots,b_n^{'}).
\end{aligned}
$$

Theorem 7. (Boundeness) Let $\{\widetilde{b}_i=[(r_i,\alpha_i),(l_i,\beta_i)]|i=1,2,\cdots,n\}$ be the set of n uncertain 2-tuple linguistic variables and $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector. then

$$
\begin{aligned}
&\Delta[min_i(r_i,\alpha_i),min_i(l_i,\beta_i)]\leqslant h^{-}\\
&\leqslant UTL-WMM^{P}(\widetilde{b}_1,\cdots,\widetilde{b}_n)\\
&\leqslant \Delta[max_i(r_i,\alpha_i),max_i(l_i,\beta_i)].
\end{aligned}
$$

Now, we will develop some special cases of UTL-WMM operator with respect to the parameter vector. Let $\{\widetilde{b}_i=[(r_i,\alpha_i),(l_i,\beta_i)]|i=1,2,\cdots,n\}$ be a collection of uncertain 2-tuple linguistic variables, $w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of $h_i(i=1,2,\cdots,n)$ with $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, and $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector.

(1) If $P=(1,0,\cdots,0)$, UTL-WMM operator will reduce to uncertain 2-tuple linguistic weighted aver-

aging operator [41]

$$UTL-WMM^{(1,0,\cdots,0)}(\widetilde{b}_1,\cdots,\widetilde{b}_n) = \Delta[(\sum_{j=1}^{n} w_j\Delta^{-1}(r_j,\alpha_j)),(\sum_{j=1}^{n} w_j\Delta^{-1}(l_j,\delta_j))]. \quad (16)$$

(2) If $P=(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})$, UTL-WMM operator will reduce to uncertain 2-tuple linguistic weighted Maclaurin symmetric mean (UTL-WMSM) operator

$$UTL-MSM^{(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})}(\widetilde{b}_1,\cdots,\widetilde{b}_n) = \Delta[(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}(\prod_{j=1}^{n}(nw_{\theta(j)}\Delta^{-1}(r_{i_j},\alpha_{i_j}))))^{\frac{1}{k}},(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}(\prod_{j=1}^{n}(nw_{\theta(j)}\Delta^{-1}(l_{i_j},\beta_{i_j}))))^{\frac{1}{k}}]. \quad (17)$$

5. Extended Uncertain 2-tuple Linguistic Weighted Muirhead Mean Operators

Herrera etc. [2] extended the 2-tuple linguistic averaging operators to accommodate the situations where the input arguments (including the attribute values and the attribute weight) are 2-tuple linguistic assessment information. Motivated by this idea, we extend UTL-WMM operator to uncertain 2-tuple linguistic weighted Muirhead mean (EUTL-WMM) operator in order to deal with some decision making problems with uncertain 2-tuple linguistic information whose attribute values are expressed in uncertain 2-tuple linguistic information and attribute weights are also represented by 2-tuple linguistic information.

Definition 13. Let $\{\widetilde{b}_i=[(r_i,\alpha_i),(l_i,\beta_i)]|i=1,2,\cdots,n\}$ be the set of n uncertain 2-tuple linguistic variables and $W=((w_1,\gamma_1),\cdots,(w_n,\gamma_n))^T$ be their associated 2-tuple linguistic weight vector, $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector. Then the uncertain 2-tuple linguistic weighted Muirhead mean operator (UTL-WMM)is defined as follows:

$$\begin{aligned}&EUTL-WMM^P(\widetilde{b}_1,\widetilde{b}_2,\cdots,\widetilde{b}_n)=UTL-WMM^P([(r_1,\alpha_1),(l_1,\beta_1)],\cdots,[(r_n,\alpha_n),(l_n,\beta_n)])\\&=\Delta[(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\frac{n\Delta^{-1}(w_{\theta(j)},\gamma_{\theta(j)})}{\sum_{j=1}^{n}\Delta^{-1}(w_j,\gamma_j)}\Delta^{-1}(r_{\theta(j)},\alpha_{\theta(j)}))^{p_j})))^{\frac{1}{\sum_{j=1}^{n}p_j}},\\&(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(\frac{n\Delta^{-1}(w_{\theta(j)},\gamma_{\theta(j)})}{\sum_{j=1}^{n}\Delta^{-1}(w_j,\gamma_j)}\Delta^{-1}(l_{\theta(j)},\beta_{\theta(j)}))^{p_j})))^{\frac{1}{\sum_{j=1}^{n}p_j}}],\end{aligned} \quad (18)$$

where $\theta(j)(j=1,2,\cdots,n)$ is any permutation of $(1,2,\cdots,n)$ and S_n is the collection of all permutations of $\theta(j)(j=1,2,\cdots,n)$.

Similar to Theorem 6 and Theorem 7, we can prove $EUTL-WMM^P(\widetilde{b}_1,\cdots,\widetilde{b}_n)$ is bounded, and monotonic.

Theorem 8. (Monotonicity) Let $\{\widetilde{b}_i=[(r_i,\alpha_i),(l_i,\beta_i)]|i=1,2,\cdots,n\}$, $\{\widetilde{b}_i^{'}=[(r_i^{'},\alpha_i^{'}),(l_i^{'},\beta_i^{'})]|i=1,2,\cdots,n\}$ be the two sets of n uncertain 2-tuple linguistic variables, $W=((w_1,\gamma_1),\cdots,(w_n,\gamma_n))^T$ be their associated 2-tuple linguistic weight vector and $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector. If $(r_i,\alpha_i)\geqslant(r_i^{'},\alpha_i^{'})$ and $(l_i,\beta_i)\geqslant(l_i^{'},\beta_i^{'})$ for any $i(i=1,2,\cdots,n)$, then

$$EUTL-WMM^P(b_1,b_2,\cdots,b_n) \geqslant EUTL-WMM^P(b_1^{'},b_2^{'},\cdots,b_n^{'}).$$

Theorem 9. (Boundeness) Let $\{\widetilde{b}_i=$

$[(r_i,\alpha_i),(l_i,\beta_i)]|i=1,2,\cdots,n\}$ be the set of n uncertain 2-tuple linguistic variables,and $W=((w_1,\gamma_1),\cdots,(w_n,\gamma_n))^T$ be their associated linguistic weight vector and $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector. then

$$\begin{aligned}&\Delta[min_i(r_i,\alpha_i),min_i(l_i,\beta_i)]\\ &\leqslant EUTL-WMM^P(\widetilde{b}_1,\cdots,\widetilde{b}_n)\\ &\leqslant \Delta[max_i(r_i,\alpha_i),max_i(l_i,\beta_i)].\end{aligned}$$

Let $\{\widetilde{b}_i=[(r_i,\alpha_i),(l_i,\beta_i)]|i=1,2,\cdots,n\}$ be a collection of uncertain 2-tuple linguistic variables, $w=(w_1,w_2,\cdots,w_n)^T$ be the 2-tuple linguistic weight vector of $h_i(i=1,2,\cdots,n)$, and $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector. Next, we will obtain some special cases of EUTL-WMM operator when the parameter takes different values.

(1) If $P=(1,0,\cdots,0)$, UTL-WMM operator will reduce to uncertain 2-tuple linguistic weighted averaging operator [19]

$$\begin{aligned}&UTL-WMM^{(1,0,\cdots,0)}(\widetilde{b}_1,\cdots,\widetilde{b}_n)\\ &=\Delta[\sum_{j=1}^{n}(\frac{\Delta^{-1}(w_j,\gamma_j)}{\sum_{j=1}^{n}\Delta^{-1}(w_j,\gamma_j)}\Delta^{-1}(r_j,\alpha_j)),\\ &\sum_{j=1}^{n}(\frac{\Delta^{-1}(w_j,\gamma_j)}{\sum_{j=1}^{n}\Delta^{-1}(w_j,\gamma_j)}\Delta^{-1}(l_j,\beta_j))],\end{aligned}\tag{19}$$

(2) If $P=(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})$, EUTL-WMM operator will reduce to uncertain 2-tuple linguistic weighted Maclaurin symmetric mean (UTL-WMSM) operator

$$\begin{aligned}UTL-MSM^{(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})}(\widetilde{b}_1,\cdots,\widetilde{b}_n)&=\Delta[(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}(\prod_{j=1}^{n}(\frac{n\Delta^{-1}(w_{\theta(j)},\gamma_{\theta(j)})}{\sum_{j=1}^{n}\Delta^{-1}(w_j,\gamma_j)}\Delta^{-1}(r_{i_j},\alpha_{i_j}))))^{\frac{1}{k}},\\ &(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}(\prod_{j=1}^{n}(\frac{n\Delta^{-1}(w_{\theta(j)},\gamma_{\theta(j)})}{\sum_{j=1}^{n}\Delta^{-1}(w_j,\gamma_j)}\Delta^{-1}(l_{i_j},\beta_{i_j}))))^{\frac{1}{k}}].\end{aligned}\tag{20}$$

6. An Approach to MAGDM with Uncertain 2-tuple Linguistic Assessment Information

In this section, we develop a multiple attribute group decision making (MAGDM) method with uncertain 2-tuple linguistic assessment information based on the proposed UTL-WMM and EUTL-WMM operator.

Suppose that a MAGDM problem has l decision makers $DM_k(k=1,2,\cdots,l)$, $A=\{A_1,A_2,\cdots,A_m\}$ is a set of m alternatives, and $C=\{C_1,C_2,\cdots,C_n\}$ is the set of attributes (or criteria). l decision makers $DM_1,\cdots,DM_l$ are given a weight vector $(\lambda_1,\cdots,\lambda_l)$ with $\lambda_i\geqslant 0$ and $\sum_{i=1}^{l}\lambda_i=1$, the weight of decision maker reflects his or her relative importance in the group decision making process. Let $D_k=(r_{ij}^k)_{m\times n}(k=1,2,\cdots,l)$ be the linguistic decision matrix of kth decision maker, where r_{ij}^k is the linguistic information provided by the kth decision maker DM_k on the assessment of A_i with respect to C_j. Let $W_k=(w_1^k,w_2^k,\cdots,w_n^k)$ be the linguistic weighted vector given by the decision maker DM_k, where w_i^k is a linguistic term assigned to attribute C_i by decision maker DM_k.

In what follows, we use UTL-WMM and EUTL-WMM operator to develop an method to solve MAGDM problems with uncertain 2-tuple linguistic assessment information. In order to obtain the best alternative(s), the following steps are involved (the decision process of proposed MAGDM method is shown as Fig. 1):

Step 1. Transform linguistic decision matrix $D_k=(r_{ij}^k)_{m\times n}$ into uncertain 2-tuple linguistic decision matrix $D_k=(\widetilde{r}_{ij}^k)_{m\times n}=([(s_{ij}^k,0),(t_{ij}^k,0)])_{m\times n}$, where $s_{ij}^k\leqslant t_{ij}^k$. Whilst, transform the lin-

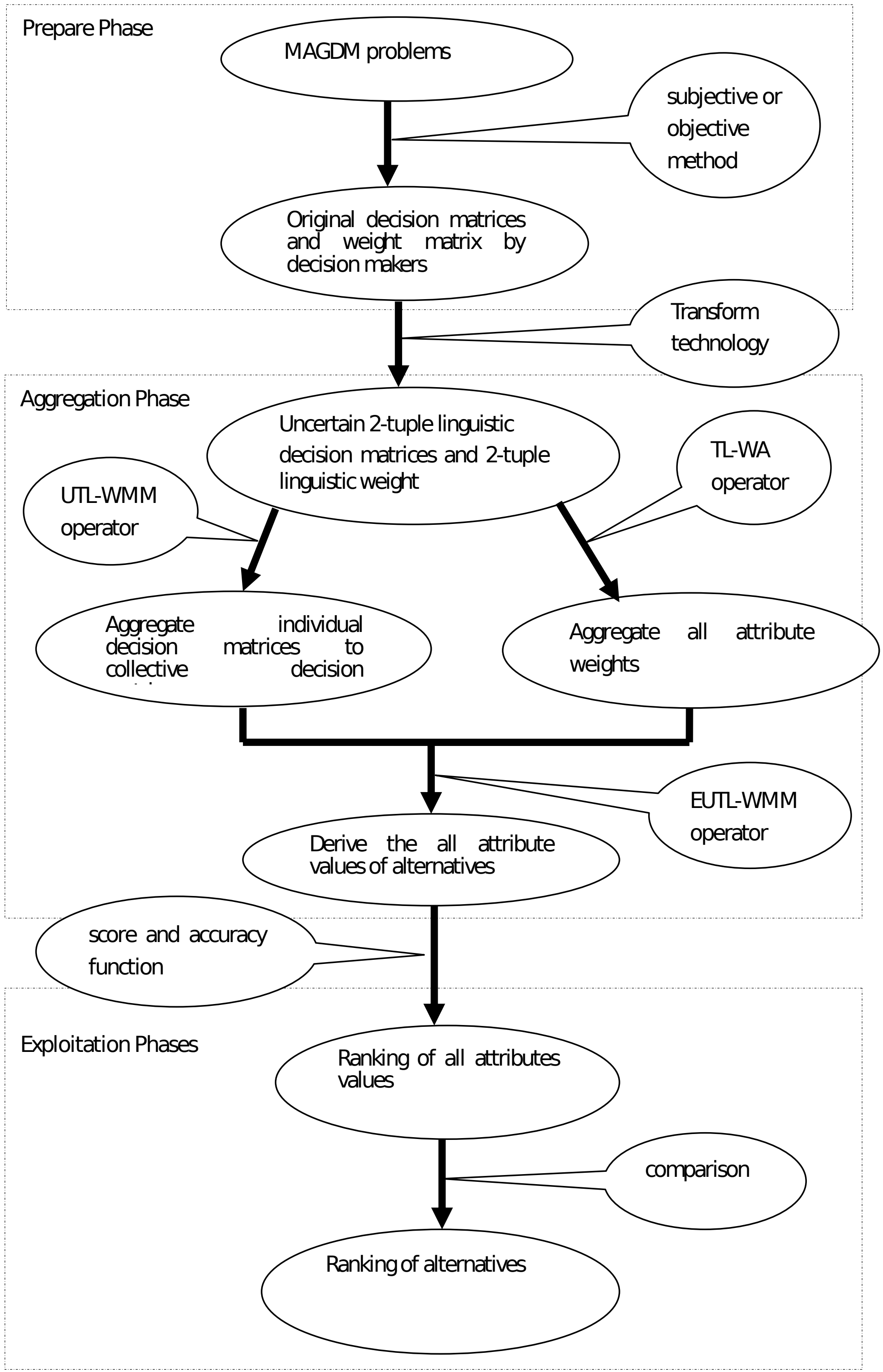

Figure 1: The decision process of proposed MAGDM methods

guistic weighted vector $W_k = (w_1^k, w_2^k, \cdots, w_n^k)$ into 2-tuple linguistic weight vector $W_k = ((w_1^k,0),(w_2^k,0),\cdots,(w_n^k,0))$.

There are three cases should be paid attention to in the process of transforming the original linguistic decision matrix into uncertain 2-tuple linguistic decision matrix. Now, we take an example to show the three cases: Let $S = \{s_0 =$ extremely poor, $s_1 =$very poor, $s_2 =$ poor, $s_3 =$ slightly poor, $s_4 =$ fair, $s_5 =$ slightly good, $s_6 =$ good, s_7= very good, $s_8 =$ extremely good $\}$. We can transform the linguistic decision matrix into uncertain 2-tuple linguistic decision matrix in the following ways:

(1) A certain grade such as good, which can be expressed as $[(s_6,0),(s_6,0)]$;

(2) A interval such as fair-good, which means the assessment of an alternative lie between poor and good, this case can be expressed as $[(s_4,0),(s_6,0)]$.

(3) If decision maker do not provide any assessment of an alternative, then the situation can be expressed as $[(s_0,0),(s_8,0)]$.

Step 2. Aggregate all individual decision matrix $D_k (k = 1,2,\cdots,l)$ to collective matrix D based on the UTL-WA operator

$$\widetilde{r}_{ij} = UTL-WA(\widetilde{r}_{ij}^1, \widetilde{r}_{ij}^2, \cdots, \widetilde{r}_{ij}^l). \tag{21}$$

Step 3. Aggregate all attribute weights provided by l decision makers based on the TL-WA operator

$$(w_j, \varepsilon_j) = \Delta[\sum_{k=1}^{l} \lambda_k \Delta^{-1}(w_j^k, 0)]. \tag{22}$$

Step 4. Utilize the EUTL-WMM operator to derive the all attribute (criteria) values $\widetilde{r}_{ij} (j = 1,2,\ldots,m)$ of the alternative A_i, i. e.

$$\widetilde{r}_i = EUTL-WMM^P(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \cdots, \widetilde{r}_{in}). \tag{23}$$

Step 5. Calculate the score values and accuracy values of $\widetilde{r}_i$ of all collective overall values

Step 6. Rank all alternatives $A_i (i = 1,2,\cdots,m)$. The bigger the $S(a_i)$, the better the A_i.

Step 7. End.

7. Numerical example and Comparative analysis

7.1. Numerical Example

In order to show the application of the proposed approach in this paper, an illustrative example was cited and adapted from [60], which an evaluation on enterprise technology innovation management. Technological innovation is not only directly related to the survival and development of an enterprise, but also affect the economic development of a region or even a country. As we all know, the management of an enterprise's technological innovation activities is an important manifestation of its technological innovation capability. In evaluating the technological innovation capability of enterprises, the following evaluation index system should be considered:

(1) G_1: Innovation system construction, attitude to innovation failure and incentives for innovation by the enterprise distribution system;

(2) G_2: Establishment and implementation of technological innovation strategy, the formation and maintenance of enterprise innovation culture;

(3) G_3: The feasibility of research and development project feasibility report;

(4) G_4: The completeness of the monitoring and evaluation system and innovation awareness of leaders and staff.

Now there are 3 decision makers DM_1, DM_2, DM_3 (weight vector $(0.3, 0.4, 0.3)$) assess the technical innovation management of 5 large enterprises $A_i (i = 1,2,\cdots,5)$ by questionnaires survey and discussion. The three decision makers employ the linguistic terms set $S = \{s_0 =$ extremely poor (EP), $s_1 =$ very poor (VP), $s_2 =$ poor (P), $s_3 =$ slightly poor (SP), $s_4 =$ Medium (M), $s_5 =$ slightly good (SG), $s_6 =$ good (G), s_7= very good (VG), $s_8 =$ extremely good (EG) $\}$ to evaluate the 5 enterprises with respect to the above evaluation criteria. The relative importance of the criteria was rated by the 3 decision makers with a set of five linguistic terms set $W = \{w_0 =$ very unimportant(VU), $w_1 =$ unimportant (U), $w_2 =$ medium (M), $w_3 =$ important (I), $w_4 =$ very important (VI)$\}$. The assessment of the five enterprises on each criteria and criteria weights provided by the three decision makers are presented in Table 1 and Table 2. Now we determin the best technology innovation management enterprise.

Now, we utilize the proposed method based on UTL-WMM and EUTL-WMM operator to drive the collective overall value, we obtain following:

Step 1. Transform original linguistic decision matrix into uncertain 2-tuple linguistic decision matrix $D_k = (\tilde{r}_{ij}^k)_{m\times n} = ([(s_{ij}^k, 0), (t_{ij}^k, 0)])_{m\times n}$ and shown in Table 3. Whilst, transform the linguistic weighted vector $W_k = (w_1^k, w_2^k, \cdots, w_n^k)$ into 2-tuple linguistic weight vector $W_k = ((w_1^k, 0), (w_2^k, 0), \cdots, (w_n^k, 0))$ and shown in Table 4.

Step 2. Aggregate all individual decision matrix $D_k (k = 1, 2, 3)$ to collective matrix D based on the UTL-WA operator and shown in Table 5.

Step 3. Aggregate all attribute weights provided by l decision makers based on the TL-WA operator and shown in Table 6,

Step 4-6. Utilize the EUTL-WMM operator to derive the all attribute (criteria) values of the alternative $A_i (i = 1, 2, \cdots, 5)$. For convenience, parameters $p = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, the aggregation results and ranking of alternatives shown in Table 7.

From Table 7, the desirable alternative is A_4.

7.2. The Influence of the Parameter Vector P on the Decision Making Results

In order to show the influence of the parameter vectors P on the decision making results, we use different parameter vectors P in our proposed methods based on HFWMM operators to rank the alternatives. The ranking results are shown in Table 2.

We explain the following aspects to illustrate the influence of parameter vector P on the decision making results:

(1) We see from the Section 3 that many uncertain 2-tuple linguistic aggregation operators are the special cases of UTL-MM and EUTL-WMM operators, so our method is more general. Specially, when $P = (\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, 0, \cdots, 0}^{k})$, the EUTL-WMM operator will become uncertain 2-tuple linguistic weighted Maclaurin mean, which is also family aggregation operators when the parameter k takes different value.

(2) It follows from Table 8 that the aggregation results obtained by EUTL-WMM operators are almost remain unchanged in this example though the parameter vector P change, this phenomenon also illustrates EUTL-WMM operators have good robust property.

(3) Parameter vector P can capture interrelationship between the individual arguments that can be fully taken into account. We can find from Table 8 that the more interrelationships of attributes which we consider, the smaller value of score functions, that is, the parameter vector P have greater control ability, the values of score function will become greater. So, different parameter vector P can be regarded as the decision makers' risk preference.

7.3. Comparisons With Other Existing Methods and Discussions

In order to verify the effectiveness of the proposed methods, we compare our proposed methods with other existing methods including the interval-valued 2-tuple VIKOR method. The results are shown in Table 9, which indicates that four methods have the same desirable alternative, which further verifies the validity of the method proposed in this paper with EUTL-WMM operator.

In the following, we will give some comparisons of the three methods and our proposed methods with respect to some characteristic, which are listed in Table 10.

IVTWA and IVTGA are two very useful aggregation operator in decision problems with interval-valued 2-tuple linguistic information. We can see from Section 3 that IVTWA and HFGA are special cases of UTL-MM operator. Compared with the method based on the IVTWA and HFGA operator, in which there are three limitations: (1)the method based on IVTWA and HFGA operator thinks that the input arguments are independent; (2) the method based on IVTWA and HFGA operator doesn't consider the interrelationship among input arguments; (3) the method based on IVTWA and HFGA only solve such a kind of decision problems in which the relative weights of attributes are evaluated in precis numerical values. Compared with the interval-valued 2-tuple linguistic VIKOR and other MAGDM methods in the literature, the proposed method has the following advantages:

(1) The proposed method has exact characteristic in linguistic information processing. It can effectively avoid the loss and distortion of information that occur formerly in the linguistic information processing.

(2) Not only the criteria of alternatives are evaluated in a linguistic manner rather than in precise numerical values, but also the weights of attributes (or criteria) are also assessed by a linguistic. It makes the DMs to express their judgments more reasonable and also makes the assessment easier to be carried out.

(3) The main advantage of these aggregation operators are that they can capture interrelationships of multiple attributes among any number of attributes by a parameter vector P and make information aggregation process more flexible by the parameter vector P.

(4) The diversity and uncertainty of DMs assessment information can be well reflected and modeled using the uncertain 2-tuple linguistic variables. It is much easier to solve the practical decision problems.

8. Conclusions

In recent years, aggregation operators play a vital role in decision making and many aggregation operators under different environment have been developed. But they still have some limitations in solving some practical problems. Some traditional Maclaurin Symmetric Mean (MSM) operator fails in dealing with the linguistic information. In this paper, we have investigated the MAGDM problems with the uncertain 2-tuple linguistic information based on new aggregation operator which can captures interrelationships of attributes among any number of attributes by a parameter vector P. To begin with, we presented some new uncertain 2-tuple linguistic MM aggregation operators to deal with MAGDM problems with uncertain 2-tuple linguistic information, including the uncertain 2-tuple linguistic Muirhead mean (UTL-MM) operator, uncertain 2-tuple linguistic weighted Muirhead mean (UTL-WMM) operator. In addition, we extend UTL-WMM operator to extended uncertain 2-tuple linguistic weighted Muirhead mean (EUTL-WMM) operators in order to deal with some decision making problems with uncertain 2-tuple linguistic information whose attribute values are expressed in uncertain 2-tuple linguistic information and attribute weight is 2-tuple linguistic information. Whilst, the some properties of these new aggregation operator were proved and some special cases were discussed. Moreover, we presented a new method to solve the MAGDM problems with uncertain 2-tuple linguistic information. Finally, we used an illustrative example to show the feasibility and validity of the new methods by comparing with the other existing methods.

In further research, it is necessary to solve the real decision making problems by applying these operators. In addition, we can develop some new aggregation operators on the basis of Muirhead mean operator by considering that MM operator has the superiority of compatibility.

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