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Multiattribute group decision-making approach with linguistic Pythagorean fuzzy information

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ABSTRACT The purpose of this study is to construct the multi-attribute group decision making (MAGDM) approach with linguistic Pythagorean fuzzy information (LPFI) based on generalized linguistic Pythagorean fuzzy aggregation operators (GLPFA). To begin with, we define the generalized indeterminacy degree-preference distance of linguistic Pythagorean fuzzy numbers (LPFNs), on the basis of it, we build a new approach for ranking the alternatives after analysing the existed comparison rule. In addition, we introduce the new version of t-norms (TNs) and t-conorms (TCs) named linguistic Pythagorean t-norms (LPTNs) and linguistic Pythagorean t-conorms (LPTCs), which can be used to handle the LPFI; some special cases for LPTNs and LPTCs are obtained and they can deal with Pythagorean fuzzy information (PFI). Thirdly, we introduce the generalized linguistic Pythagorean fuzzy average aggregation operator (GLPFAA) based on LPTN and LPTC along with their properties are also investigated, whilst, some special cases of GLPAA are obtained when LPTN and LPTC take some special TNs and TCs. Finally, a MAGDM approach based on some LPTNs and LPTCs is constructed to deal with some MAGDM problems with unknown attributes'weights and experts' weights, before building the MAGDM approach, we define new cross-entropy to fix the experts's weights and use the maximizing deviation to calculate the attributes' weights based on the proposed indeterminacy degree-preference distance. Consequently, an illustrative example is provided in order to show the effectiveness and advantages of the proposed method and some comparisons are also carried out.

INDEX TERMS Linguistic Pythagorean fuzzy set (LPFS), Linguistic Pythagorean t-norms (LPTNs), Linguistic Pythagorean t-conorms (LPTCs), Generalized linguistic Pythagorean aggregation operators, Generalized indeterminacy degree-preference distance, Generalized linguistic Pythagorean cross-entropy.

I. INTRODUCTION

THe theories and methods of multi-attribute decision making (MADM) are widely used in different fields such as economy, management and engineering. MADM uses decision information to rank all limited projects and select the best ones through certain ways. Because of the complexity and uncertainty of objective things and people's fuzzy thinking, it is often difficult for people to give precise figures in the decision-making process, and attribute values appear in the form of fuzzy information. Among these fuzzy information, intuitionistic fuzzy set (IFS) [1] is considered to be more appropriate to represent and process imprecise, uncertain and vague information in some decision making problems (DMPs). Since IFS' appearance, the theories and applications on IFS are all comprehensively studied. An important application field is fuzzy decision making, although IFS has been successfully applied in some multi-attribute decision making (MADM), the sum of membership degree (MD) and non-membership degree (NMD) may be greater than 1 in some special real decision problems, this situation could not be described by IFS. In order to address some DMPs, Pythagorean fuzzy set (PFS), an important extension of IFS, proposed by Yager [2]. PFS is also characterized by MD and NMD and the square sum of MD and NMD is also less than 1, but sum of them may be more than 1. Since PFS's appearance, theory of PFS and its applications have been studied in depth, for example some information measures [3]–[5], improved score function [6]–[9], new operational laws [10], [11], aggregation operators [12]–[16] and many decision-making approaches [17], [18]. These studies are limited to deal with some uncertain information in quantitative environments. However, in some real decision-making environments, the optimal expression of imprecise information and uncertainty naturally presents the form of linguistic terms owing to the complexity of the problem and the inherent fuzziness of human preferences. However, sometimes in real life, DMP is presented by expressing the qualitative aspects of uncertainties and inaccurate information. In this case, decision makers often use linguistic variables [19], [49] to give their opinions on alternatives. In order to describe it, Xu [20], [21] proposed the linguistic term set (LTS) and continuous linguistic term set (CLTS) and also investigated the linguistic aggregation operators. Since the LTS's appearance, some extended linguistic fuzzy sets have been established and applied to some DMPs, the extended LTSs are mainly focus on the following three aspects: (1) Linguistic hesitant fuzzy set (LHFS): Rodriguez [22] introduced the hesitant linguistic terms set (HLTSs) and built a linguistic decision-making model in which experts give their opinions by eliciting linguistic expressions. Liao [23] built qualitative decision making approach with correlation coefficients of HLTSs; Wei [24] defined some uncertain measures in hesitant linguistic environment and applied to MADM; Gou [25], [26] established the related information measures of HLTSs and related decision making approaches based on these information measures; Liu [27] defined the distance measures for HLTSs and applied them to MADM; Zhou [28] established MCDM approaches based on distance measures for LHFSs; Yang [29] introduced cross-entropy measures under the linguistic hesitant intuitionistic environment; Farhadinia [30] defined some information measures under hesitant linguistic environment and MADM approach. (2) Linguistic neutrosophic sets: Li [31] introduced linguistic neutrosophic sets (LNSs)and MCDM approach is also built based on two aggregation operators under linguistic neutrosophic environments; Li [32] developed MCGDM approach with EDAs method based on LNNSs. (3) Linguistic intuitionistic fuzzy sets (LIFS): Zhang [33] defined the LIFS by combing linguistic approach and IFS in which decision maker expressed the MD and NMD by the linguistic terms. Chen [34] established an approach to MADM based on LINSs; Garg [35] introduced some aggregation operators for LIFS by using the set pair analysis theory; Liu [36], [37] defined scaled prioritized operators and power Bonferroni operators based on LIFNs and applied them to MADM; Liu [38] proposed a new approach to MADM With LI Information based on Dempster-Shafer evidence theory; Zhang [57] established the outranking method for MCDM with LIFNs. (4) Linguistic Pythagorean fuzzy sets(LPFSs): rencently, Garg [40] introduced the concept of LPFSs based on LIFSs, some aggregation operators are also defined and MAGDM approach with linguistic Pythagorean fuzzy information (LPFI) is built based on the proposed aggregation operators.

The research motivations of the present work can be summarized as following: aggregation operator as a useful tool to aggregate relevant information has been focused and also used in many DMPs. There are many kinds of aggregation operators [11]–[15], [35]–[37], [41], [42], [59]–[61] to deal with IFNs, PFNs or LPFNs. Of course, the most important aspect of aggregation operators is to build operational laws which is on the basis of t-norms (TNs) and t-conorms (TCs). Obviously, the above mentioned aggregation operators are only obtained by the algebraic TN and TCs, which are just a kind of TNs and TCs. The Archimedean TNs and TCs are the generalization of various TNs and TCs, respectively, which provide some very useful special cases of operations, such as Algebraic operations, Einstein operations and Hamacher operations, Frank operations, Dombi operations [43] and so on. Some aggregation operators based on these generalized operational laws under some fuzzy environments are studied, for instance, Zhang [44] introduced interval-valued intuitionistic fuzzy Frank aggregation operators and applied to MADM; Jana [45] introduced picture fuzzy Dombi aggregation operators and applied to MADM. However, it is a pity that the Archimedean TNs and TCs are restricted to[0, 1], they are suitable for dealing with IFNs, but they cannot be used to aggregate the some linguistic information. Tao [46] extended TNs and TCs from [0, 1] to [0, t] for aggregating interval linguistic labels; Liu [47] also extended TNs and TCs from [0, 1] to [0, t] to deal with intuitionistic 2-tuple linguistic information (I2LI). However, these (extended) TNs and TCs can not be used in aggregating LPFI.

This paper focuses on developing MAGDM approach based on LPFI with unknown experts weights and attributes weights. The goals and contributions of this work are:

(1) to give new approach to rank LPFNs with the help of academical thoughts of [48].

(2) to extend the range [0, 1] of the Archimedean TNs and TCs into [0, t] ($t > 0$)and propose the new version of TNs and TCs which defined in [0, t] to deal with the LPI, specially, namely, linguistic Pythagorean TNs (LPTNs) and linguistic Pythagorean TCs (LPFCs) which can handle Pythagorean fuzzy information.

(3) to propose some new general operational laws for the LPFI and to propose the generalized linguistic Pythagorean aggregation (GLPFWA) operator for the LPFI.

(4) to propose generalized indeterminacy degree-preference distance and generalized linguistic Pythagorean fuzzy cross-entropy to fix the expert's weights and attribute weights, respectively.

(5) to construct a novel MAGDM approach with the LPFI based on the proposed GLPFWA under the expert's weights and attribute weights are unknown.

For the sake of the above objectives, the organizational structure of this paper is as follows. We firstly review some definitions on linguistic term set (LTS), linguistic Pythagorean fuzzy sets (LPFSs) in Section 2. Section 3 is devoted to the new approach for ranking the alternatives based on generalized hesitant degree-preference distance under LPFI. Section 4 is focused on new version of TNs and TCs named linguistic Pythagorean TNs (LPTNs) and linguistic Pythagorean TCs (LPTCs), respectively, which can handle the LPFI, and some special cases for LPTNs and LPTCs are obtained and they can deal with PFI. In Section 5, we introduce the GLPFWA based on LPTN and LPTC along with their properties are also investigated. In Section 6, we analyse the GLPFWA and some special cases are obtained when LPTN and LPTC take some special TNs and TCs, respectively, and some parameters changed. Section 7 is devoted to construct a MAGDM approach based on some LPTNs and LPTCs, before building the MAGDM approach, we define new cross-entropy to fix the experts's weights and use the maximizing deviation to calculate the attributes' weights based on the generalized indeterminacy degree-preference distance proposed in Section 3. Consequently, a practical example is provided in Section 8 to reveal the effectiveness and advantages of the proposed method and some conclusions of this study are made in Section 9.

II. PRELIMINARIES

Some basic concepts of PFSs and LPFSs will be reviewed in this part, which are the basis of the present work.

A. LINGUISTIC FUZZY SET

Let $\mathfrak{L} = \{s_i | i = 0, 1, \cdots, g\}$ be a LTS with odd cardinality, for any label s_i, which stands for a possible value for a linguistic variable and satisfies the following condition [49]:

(1)$s_i > s_j \Leftrightarrow i > j$;
(2) when $s_i \geq s_j$, then $max\,(s_i, s_j) = s_i$;
(3) when $s_i \geq s_j$, then $min\,(s_i, s_j) = s_j$;
(4) $Neg\,(s_i) = s_j$, where $j = g - i$.

Later, Xu [20] defined a continuous linguistic term set (CLTS) $\bar{\mathfrak{S}} = \{s_a | s_0 \leq s_a \leq s_g, a \in [0, g]\}$ by adding the virtual term. If $s_a \notin \bar{\mathfrak{S}}$, then s_a is called the virtual term, otherwise, it is called original term.

Garg [40] introduced the concept of LPFS. We will review the PFS before review the LPFS.

B. PYTHAGOREAN FUZZY SET AND LINGUISTIC PYTHAGOREAN FUZZY SET

Definition 1: [2] Let $X = \{\Psi_1, \Psi_2, \cdots, \Psi_n\}$ be a finite universe of discourse, an Pythagorean fuzzy set (PFS) A in X is defined as

$$A = \{\langle x, (\mu_A(x), \nu_A(x))\rangle | x \in X\}.$$

where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ with the condition $0 \leq \mu_A^2(x) + \nu_A^2(x) \leq 1$, μ_A, ν_A are called membership function, and non-membership function, respectively. $\pi_A(x) = \sqrt{1 - \mu_A^2(x) - \nu_A^2(x)}$ represents the degree of indeterminacy of x to A and called the indeterminacy degree. For simplicity, called $(\mu_A(x), \nu_A(x))$ is an Pythagorean fuzzy number (PFN) and denoted by (μ_A, ν_A).

Definition 2: [40] Let $X = \{\Psi_1, \Psi_2, \cdots, \Psi_n\}$ be a finite universe of discourse and $\bar{\mathfrak{S}} = \{s_\alpha | s_0 \leq s_\alpha \leq s_t, \alpha \in [0, t]\}$, be a CLTS. A LPFS A is defined as follows:

$$A = \{(s_u(x), s_v(x)) | x \in X\}$$

where $x_u(x), s_v(x) \in \bar{\mathfrak{S}}$ with the condition $u^2 + v^2 \leq t^2$ and they are called linguistic membership degree (LMD) and linguistic nonmembership degree (LNMD) of x to A, respectively. $\pi_A(x) = s_{\sqrt{t^2 - u^2 - v^2}}$ is called the degree of linguistic indeterminacy. For simplicity, denote $(s_u(x), s_v(x))$ as $A = (s_u, s_v)$ and called as linguistic Pythagorean fuzzy number (LPFN).

For any LPFN $A = (s_{u_1}, s_{v_1})$, the complement of A is $A^c = (s_{v_1}, s_{u_1})$.

III. COMPARISON METHOD OF LPFNS

Given a finite set of alternatives, a linguistic Pythagorean fuzzy MADM problem is a kind of problem in which the evaluation of each alternative w. r. t. a set of attributes is expressed by LPFNs, and the most desirable alternative is selected based on the degree of suitability to which each alternative meets the the requirements of decision-makers. However, the size relations or the inclusion relations does not exist in LPFS under ambient conditions, some comparison technologies of LPFNs have been developed to determine the order relations of LPFNs. As an important tool to compare LPFNs in order to get the desirable one in DMPs, score function is needed to convert LPFNs into real numbers in order to become easier to compare with each other in the process of decision making.

- **Comparison Rule I**

Definition 3: [40] Let $A = (s_{u_i}, s_{v_i})$ be a LPFN with $s_{u_i}, s_{v_i} \in \bar{\mathfrak{S}}$. The score function of A is given as following

$$\mathcal{S}(A) = s_{\sqrt{(t^2 + u_i^2 - v_i^2)/2}} \tag{1}$$

and the accuracy function is defined as

$$\mathcal{H}(A) = s_{\sqrt{(u_i^2 + v_i^2)/2}} \tag{2}$$

The comparison rule is also defined as follows by Garg [40] based on Eq.(1) and Eq.(2):

(1) If $\mathcal{S}(A) > \mathcal{S}(B)$, then $A \succ B$;
(2) If $\mathcal{S}(A) = \mathcal{S}(B)$ and
(2.1) $\mathcal{H}(A) = \mathcal{H}(B)$, then $A = B$;
(2.2) $\mathcal{H}(A) > \mathcal{H}(B)$, then $A \succ B$.

However, there are obvious neglectable shortcomings in the above-mentioned LPFN comparison methods. The scoring function or accuracy function can not be used separately to obtain the final order of alternatives. On the contrary, it needs to combine $\mathcal{S}(A)$ and $\mathcal{H}(A)$ to compare different LPFNs, which leads to a loss of consistency to some extent. In addition, the above-mentioned comparison method can not consider the object influence of indeterminacy degree for some special LPFNs, nor can it produce reasonable ranking. Now we show this point by taking an example from Garg [40].

Example 1: Let $A = (s_3, s_1), B = (s_5, s_3), C = (s_4, s_4), D = (s_2, s_0)$ be four LPFNs which are from $\bar{S} = \{s_a | s_0 \leq s_a \leq s_6 | a \in [0,6]\}$, by above comparison rule, it follows that

$\mathcal{S}(A) = s_{4.69}, \mathcal{S}(B) = s_{5.10}, \mathcal{S}(C) = s_{4.24}, \mathcal{S}(D) = s_{4.47}$.

Therefore, $B \succ A \succ D \succ C$.

• **Comparison Rule II**

The above comparison rule obtained that $D \succ C$. Although $D \succ C$, the complete information contained in $C \succ D$. Therefore, it is unreasonable to draw a conclusion that $D \succ C$. Hence, a more suitable comparison rule needs to be studied to compare two LPFNs. Szmidt et al [48] proposed an effective sorting method for IFNs by comprehensively accounting for hesitancy degree of IFNs and the distance between the IFN and the positive ideal solution (PIS). With the help of the academic thoughts of [48], an effective comparison method will be developed for compare two LPFNs, and above listed drawbacks will be overcame. In order to introduce the new ranking method, the distance measure of two LPFNs will be proposed.

However, in practice, decision makers usually have different preferences for different distance measurements. Therefore, the distance measure with preference information between any two LPFNs can be defined. For any two LPFNs, the generalized indeterminacy degree-preference distance can be defined as:

Definition 4: Let $A = (s_{u_1}, s_{v_1})$ and $B = (s_{u_2}, s_{v_2})$ be two LPFNs, which are derived from $\bar{S} = \{s_a | s_0 \leq s_a \leq s_t, a \in [0,t]\}$. Then the generalized indeterminacy degree-preference distance of A and B is defined as:

$$d(A,B) = \left\{(1-p)\left[\left|\left(\frac{u_1}{t}\right)^2 - \left(\frac{u_2}{t}\right)^2\right|^\lambda + \left|\left(\frac{v_1}{t}\right)^2 - \left(\frac{v_2}{t}\right)^2\right|^\lambda\right] + p\left|\left(\frac{\sqrt{t^2-u_1^2-v_1^2}}{t}\right)^2 - \left(\frac{\sqrt{t^2-u_2^2-v_2^2}}{t}\right)^2\right|^\lambda\right\}^{\frac{1}{\lambda}}, \quad (3)$$

where $\lambda > 0$ and $p \in [0,1]$.

We will obtain some different distance measures when parameters λ, p take different values.

Case 1. When $\lambda = 1$, the generalized indeterminacy degree-preference distance will reduce to Hamming-indeterminacy degree-preference distance

$$d(A,B) = \frac{1-p}{2}\left[\left|\left(\frac{u_1}{t}\right)^2 - \left(\frac{u_2}{t}\right)^2\right| + \left|\left(\frac{v_1}{t}\right)^2 - \left(\frac{v_2}{t}\right)^2\right|\right] + p\left|\left(\frac{\sqrt{t^2-u_1^2-v_1^2}}{t}\right)^2 - \left(\frac{\sqrt{t^2-u_2^2-v_2^2}}{t}\right)^2\right|.$$

In **Case 1**, if $p = 0$, which means the influence of indeterminacy degree was not taken into account. The generalized indeterminacy degree-preference distance reduce to metric distance

$$d(A,B) = \frac{1}{2}\left[\left|\left(\frac{u_1}{t}\right)^2 - \left(\frac{u_2}{t}\right)^2\right| + \left|\left(\frac{v_1}{t}\right)^2 - \left(\frac{v_2}{t}\right)^2\right|\right].$$

Case 2. When $\lambda = 2$, the generalized indeterminacy degree-preference distance will reduce to Euclidean-indeterminacy degree-preference distance

$$d(A,B) = \left\{(1-p)\left[\left|\left(\frac{u_1}{t}\right)^2 - \left(\frac{u_2}{t}\right)^2\right|^2 + \left|\left(\frac{v_1}{t}\right)^2 - \left(\frac{v_2}{t}\right)^2\right|^2\right] + p\left|\left(\frac{\sqrt{t^2-u_1^2-v_1^2}}{t}\right)^2 - \left(\frac{\sqrt{t^2-u_2^2-v_2^2}}{t}\right)^2\right|^2\right\}^{\frac{1}{2}}.$$

In **Case 2**, If $p = 0$, the generalized indeterminacy degree-preference distance will reduce to Euclidean distance

$$d(A,B) = \left\{\frac{1}{2}\left[\left|\left(\frac{u_1}{t}\right)^2 - \left(\frac{u_2}{t}\right)^2\right|^2 + \left|\left(\frac{v_1}{t}\right)^2 - \left(\frac{v_2}{t}\right)^2\right|^2\right]\right\}^{\frac{1}{2}}.$$

Let A, B be two LPFNs. It is easy to verify that generalized indeterminacy degree-preference distance d satisfies the following properties:

(1) $d(A,B) \geq 0$;
(2) $d(A,B) = d(B,A)$;
(3) $d(A,B) = 0 \Leftrightarrow A = B$.

Now, we introduce a new comparison method based on above mentioned distance.

Definition 5: Let $A = (s_u, s_v)$ be a LPFN, which is derived from $\bar{\mathfrak{S}} = \{s_a | s_0 \leq s_a \leq s_t, a \in [0,t]\}$. $P = (s_t, s_0)$ is the positive ideal point. Then, the ordering index R of A is defined as:

$$R(A) = \frac{1}{2}\left(1 + \frac{\sqrt{t^2-u^2-v^2}}{t}\right) d(A,P). \quad (4)$$

Where $d(A,P)$ is the generalized indeterminacy degree-preference distance.

It is obvious that the lower the value of $R(A)$ in Eq.(4), the better the alternative A from the view of point that the amount of positive information include, and reliability of information.

Example 2: Let $A = (s_5, s_2), B = (s_3, s_4), C = (s_6, s_1), D = (s_4, s_3)$ be four LPFNs which are from $\bar{S} = \{s_a | s_0 \leq s_a \leq s_8 | a \in [0,8]\}$, then,

(1) According to comparison rule I, we have $\mathcal{S}(A) = s_{6.5192}, \mathcal{S}(B) = s_{5.3385}, \mathcal{S}(C) = s_{7.0356}, \mathcal{S}(D) = s_{5.9582}$, therefore $C \succ A \succ D \succ B$.

(2) When $p = 0.5, \Psi = 1$ in the Eq.(3), according to comparison rule II, we have $R(A) = 0.6165, R(B) = 0.8885, R(C) = 0.3715, R(D) = 0.8013$, therefore $C \succ A \succ D \succ B$.

Example 3: Let $A = (s_3, s_1), B = (s_5, s_3), C = (s_4, s_4), D = (s_2, s_0)$ be four LPFNs which are from $\bar{S} = \{s_a | s_0 \leq s_a \leq s_6 | a \in [0,6]\}$, by above comparison, it follows that

$R(A) = 0.6342, R(B) = 0.1488, R(C) = 0.2829, R(D) = 0.8395$.

Therefore, $B \succ C \succ A \succ D$, which is more reasonable than the result of Example 1.

It follows from Definition 5 that the new ordering function focuses not only on the information we really have, but also on the lack of information, because the two aspects both influence the ranking of alternatives.

IV. LINGUISTIC PYTHAGOREAN TN AND LINGUISTIC PYTHAGOREAN TC

TN and TC are widely used in the aggregation operators. However, the domain and the range of TN and TC must be in $[0,1]$. If the LMD and LNMD of a LPFN can be converted into $[0,t]$, where $t>0$, then all operations can be done in $[0,t]$. Therefore, in order to investigate I2LI, Liu [47] introduced the concepts of extended TN and extended TC:

Definition 6: [47] An extended TN is a function $\mathcal{T}_N:[0,t]^2\rightarrow[0,t]$ which satisfies the conditions: for any $a,b,c,a_1,b_1\in[0,t]$,

(1) $\mathcal{T}_N(a,0)=0, \mathcal{T}_N(a,t)=a$;
(2) $\mathcal{T}_N(a,b)=\mathcal{T}_N(b,a)$;
(3) $\mathcal{T}_N(a,\mathcal{T}_N(b,c))=\mathcal{T}_N(\mathcal{T}_N(a,b),c)$;
(4) if $a\le a_1, b\le b_1$, then $\mathcal{T}_N(a,b)\le\mathcal{T}_N(a_1,b_1)$

The extended TN $\mathcal{T}_N$ has the following properties:
(1) $\mathcal{T}_N(a,b)$ is continuous,
(2) $\mathcal{T}_N(a,b)\le min\{a,b\}$.

Definition 7: [47] An extended TC is a function $\mathcal{T}_C:[0,t]^2\rightarrow[0,t]$ which satisfies the conditions: for any $a,b,c,a_1,b_1\in[0,t]$,

(1) $\mathcal{T}_C(a,0)=a, \mathcal{T}_C(a,t)=t$;
(2) $\mathcal{T}_C(a,b)=\mathcal{T}_C(b,a)$;
(3)$\mathcal{T}_C(a,\mathcal{T}_C(b,c))=\mathcal{T}_C(\mathcal{T}_C(a,b),c)$;
(4)if $a\le a_1, b\le b_1$, then $\mathcal{T}_C(a,b)\le\mathcal{T}_C(a_1,b_1)$

The extended TC $\mathcal{T}_C$ has the following properties:
(1) $\mathcal{T}_C(a,b)$ is continuous,
(2) $\mathcal{T}_C(a,b)\ge max\{a,b\}$.

With the help of extended TN and TC, we will investigate some linguistic Pythagorean TNs (LPTNs) and linguistic Pythagorean TCs (LPTCs) which can deal with LPFI and PFI.

Liu [47] pointed out that the following condition should be satisfied for their generators:

As far as the extended TN is concerned, a monotonically decreasing function ξ called a generator of extended TN, if it satisfies $\xi:[0,t]\rightarrow\mathbf{R}^+$ and $\xi^{-1}:\mathbf{R}^+\rightarrow[0,t]$ with $lim_{a\rightarrow\infty}\xi^{-1}(a)=0$ and $\xi^{-1}(0)=t$, where $a\in[0,t]$. According to Dombi [43], the extended TN $\mathcal{T}_N(a,b)=\xi^{-1}(\xi(a)+\xi(b))$.

As far as the extended TC is concerned, a monotonically increasing function ζ called a generator of extended TC, if it satisfies $\zeta:[0,t]\rightarrow\mathbf{R}^+$ and $\zeta^{-1}:\mathbf{R}^+\rightarrow[0,t]$ with $lim_{a\rightarrow\infty}\zeta^{-1}(a)=t$ and $\zeta^{-1}(0)=0$, where $a\in[0,t]$. According to Dombi [43], the extended TC $S(a,b)=\zeta^{-1}(\zeta(a)+\zeta(b))$. For the relation of generator of LPTNs and LPTCs can be given as the following equation:

$$\zeta(a)=\xi\left(\sqrt{t^2-a^2}\right).$$

Now, according to the generators, we introduce LPTNs and LPTCs which can deal with LPFI.

• Let generator of the linguistic Pythagorean algebraic TN $\mathcal{T}_N^A$ be $\xi(a)=-ln\left(\frac{a}{t}\right)^2$, where $\xi^{-1}(a)=te^{-\frac{1}{2}a}$ and the generator of the linguistic Pythagorean algebraic TC $\mathcal{T}_C^A$ be $\zeta(a)=-ln\frac{t^2-a^2}{t^2}$, where $\zeta^{-1}(a)=\left(t^2-t^2e^{-a}\right)^{\frac{1}{2}}$.

According to the generators of the $\mathcal{T}_N^A$ and $\mathcal{T}_N^A$, we have

Theorem 1: For any $a,b\in[0,t]$, the $\mathcal{T}_N^A$ and $\mathcal{T}_N^A$ can be described by

$$\mathcal{T}_N^A(a,b)=\frac{ab}{t},$$
$$\mathcal{T}_C^A(a,b)=\sqrt{(a^2+b^2)-\frac{a^2b^2}{t^2}}.$$

Proof As $\xi(a)+\xi(b)=-ln\left(\left(\frac{a}{t}\right)^2\left(\frac{b}{t}\right)^2\right)=-ln\frac{a^2b^2}{t^4}$, so we have

$$\begin{aligned}\mathcal{T}_N^A(a,b) &= \xi^{-1}(\xi(a)+\xi(b))=t\left(e^{-(\xi(a)+\xi(b))}\right)^{\frac{1}{2}}\\ &= t\left(e^{ln\frac{a^2b^2}{t^4}}\right)^{\frac{1}{2}}=t\left(\frac{a^2b^2}{t^4}\right)^{\frac{1}{2}}=\frac{ab}{t}.\end{aligned}$$

Similarly, as $\zeta(a)+\zeta(b)=-ln\left(\left(\frac{t^2-a^2}{t^2}\right)\left(\frac{t^2-b^2}{t^2}\right)\right)-ln\frac{(t^2-a^2)(t^2-b^2)}{t^4}$, so we have

$$\begin{aligned}\mathcal{T}_C^A(a,b) &= \zeta^{-1}(\zeta(a)+\zeta(b))\\ &= t\left(1-e^{-(\zeta(a)+\zeta(b))}\right)^{\frac{1}{2}}\\ &= t\left(1-e^{ln\frac{(t^2-a^2)(t^2-b^2)}{t^4}}\right)^{\frac{1}{2}}\\ &= t\left(1-\frac{(t^2-a^2)(t^2-b^2)}{t^4}\right)^{\frac{1}{2}}\\ &= \left(t^2-\frac{(t^2-a^2)(t^2-b^2)}{t^2}\right)^{\frac{1}{2}}\\ &= \left((a^2+b^2)-\frac{a^2b^2}{t^2}\right)^{\frac{1}{2}}.\end{aligned}$$

In Theorem 1, if $t=1$, then the TN $\mathcal{T}_N^A(a,b)$ and TC $\mathcal{T}_C^A(a,b)$ under the linguistic Pythagorean fuzzy environment will reduce to Pythagorean t-norm $T(a,b)$ and Pythagorean t-conorm $S(a,b)$

$$T(a,b)=ab,\ \ S(a,b)=\sqrt{(a^2+b^2)-a^2b^2}.$$

• Let generator of the linguistic Pythagorean Einstein TN $\mathcal{T}_N^E$ be $\xi(a)=ln\left(\frac{2t^2-a^2}{a^2}\right)$, where $\xi^{-1}(a)=\frac{\sqrt{2}t}{\sqrt{1+e^a}}$ and the generator of the linguistic Pythagorean Einstein TC $\mathcal{T}_C^E$ be $\zeta(a)=ln\frac{t^2+a^2}{t^2-a^2}$, where $\zeta^{-1}(a)=t\sqrt{\frac{e^a-1}{e^a+1}}$.

According to the generators of the $\mathcal{T}_N^E$ and $\mathcal{T}_C^E$, we have

Theorem 2: For any $a, b \in [0, t]$, the $\mathcal{T}_N^E$ and $\mathcal{T}_C^E$ can be described by

$$\mathcal{T}_N^E(a,b) = \frac{\sqrt{2}tab}{\sqrt{a^2b^2 + (2t^2-a^2)(2t^2-b^2)}},$$
$$\mathcal{T}_C^E(a,b) = t^2\sqrt{\frac{a^2+b^2}{t^4+a^2b^2}}.$$

Proof As $\xi(a) + \xi(b) = ln\left(\frac{2t^2-a^2}{a^2}\cdot\frac{2t^2-b^2}{b^2}\right) = ln\frac{(2t^2-a^2)(2t^2-b^2)}{a^2b^2}$, so we have

$$\begin{aligned}\mathcal{T}_N^E(a,b) &= \xi^{-1}(\xi(a)+\xi(b)) \\ &= \frac{\sqrt{2}t}{\sqrt{1+e^{\xi(a)+\xi(b)}}} \\ &= \frac{\sqrt{2}t}{\sqrt{1+\frac{(2t^2-a^2)(2t^2-b^2)}{a^2b^2}}} \\ &= \frac{\sqrt{2}tab}{\sqrt{a^2b^2+(2t^2-a^2)(2t^2-b^2)}}.\end{aligned}$$

Similarly, as $\zeta(a)+\zeta(b) = ln\left(\frac{t^2+a^2}{t^2-a^2}\right)\left(\frac{t^2+b^2}{t^2-b^2}\right) = ln\frac{(t^2+a^2)(t^2+b^2)}{(t^2-a^2)(t^2-b^2)}$, so we have

$$\begin{aligned}\mathcal{T}_C^E(a,b) &= \zeta^{-1}(\zeta(a)+\zeta(b)) \\ &= t\sqrt{\frac{e^{\zeta(a)+\zeta(b)}-1}{e^{\zeta(a)+\zeta(b)}+1}} \\ &= t\sqrt{\frac{e^{ln\frac{(t^2+a^2)(t^2+a^2)}{(t^2-a^2)(t^2-b^2)}}-1}{e^{ln\frac{(t^2+a^2)(t^2+b^2)}{(t^2-a^2)(t^2-b^2)}}+1}} \\ &= t\sqrt{\frac{\frac{(t^2+a^2)(t^2+b^2)}{(t^2-a^2)(t^2-b^2)}-1}{\frac{(t^2+a^2)(t^2+b^2)}{(t^2-a^2)(t^2-b^2)}+1}} \\ &= t^2\sqrt{\frac{a^2+b^2}{t^4+a^2b^2}}.\end{aligned}$$

In Theorem 2, if $t = 1$, then the $\mathcal{T}_N^E$ and $\mathcal{T}_C^E$ will reduce to t-norm T_ϵ and t-conorm S_ϵ under the Pythagorean fuzzy environment

$$T_\epsilon(a,b) = \frac{ab}{\sqrt{1+(1-a^2)(1-b^2)}}, \quad S_\epsilon(a,b) = \sqrt{\frac{a^2+b^2}{1+a^2b^2}}.$$

- Let generator of the linguistic Pythagorean Hamacher TN $\mathcal{T}_N^H$ be $\xi(a) = ln\left(\frac{\gamma t^2+(1-\gamma)a^2}{a^2}\right)$, where $\xi^{-1}(a) = \frac{\sqrt{\gamma}t}{\sqrt{e^a+\gamma-1}}$ and the generator of the linguistic Pythagorean Hamacher TC $\mathcal{T}_C^H$ be $\zeta(a) = ln\frac{\gamma t^2+(1-\gamma)(t^2-a^2)}{t^2-a^2}$, where $\zeta^{-1}(a) = t\sqrt{\frac{e^a-1}{e^a+\gamma-1}}$, where $\gamma > 0$.

According to the generators of the $\mathcal{T}_N^H$ and $\mathcal{T}_C^H$, similar to proof of Theorem 2, we have

Theorem 3: For any $a, b \in [0, t]$, the the $\mathcal{T}_N^H$ and $\mathcal{T}_C^H$ can be described by Eq. (5) and Eq. (6).

In Theorem 3, if $t = 1$, then $\mathcal{T}_N^H$ and $\mathcal{T}_C^H$ will reduce to t-norm $T_{\mathcal{H}}(x,y)$ and t-conorm $S_{\mathcal{H}}(a,b)$ under the Pythagorean fuzzy environment

$$T_H(a,b) = \frac{\sqrt{\gamma}ab}{\sqrt{(\gamma-1)a^2b^2+(\gamma-(1-\gamma)a^2)(\gamma-(1-\gamma)b^2)}}$$
$$S_H(a,b) = \sqrt{\frac{(a^2+b^2)+(\gamma-2)a^2b^2}{1+(\gamma-1)a^2b^2}}.$$

If $\gamma = 1$ in Theorem 3, $\mathcal{T}_N^H$ and $\mathcal{T}_C^H$ will be reduced to $\mathcal{T}_N^A$ and $\mathcal{T}_C^A$, respectively. If $\gamma = 2$ in Theorem 3, $\mathcal{T}_N^H$ and $\mathcal{T}_C^H$ will reduce to $\mathcal{T}_N^E$ and $\mathcal{T}_C^E$, respectively.

- Let generator of the linguistic Pythagorean Frank TN $\mathcal{T}_N^F$ be $\xi(a) = ln\left(\frac{\gamma-1}{\gamma^{\frac{a^2}{t^2}}-1}\right)$, where $\xi^{-1}(a) = t\sqrt{log_\gamma\left(\frac{e^a+(\gamma-1)}{e^a}\right)}$ and the generator of the linguistic Pythagorean Frank t-conorm $\mathcal{T}_C^F$ be $\zeta(a) = ln\left(\frac{\gamma-1}{\gamma^{1-\frac{a^2}{t^2}}-1}\right)$, where $\zeta^{-1}(a) = t\sqrt{log_\gamma\left(\frac{\gamma e^a}{e^a+\gamma-1}\right)}$, where $\gamma > 1$.

According to the generators of $\mathcal{T}_N^F$ and $\mathcal{T}_C^F$, similar to proof of Theorem 2, we have

Theorem 4: For any $a, b \in [0, t]$, the $\mathcal{T}_N^F$ and $\mathcal{T}_C^F$ can be described by

$$\mathcal{T}_N^F(a,b) = t\sqrt{log_\gamma\frac{(\gamma-1)+\left(\gamma^{\frac{a^2}{t^2}}-1\right)\left(\gamma^{\frac{b^2}{t^2}}-1\right)}{\gamma-1}},$$
$$\mathcal{T}_C^F(a,b) = t\sqrt{log_\gamma\frac{\gamma(\gamma-1)}{(\gamma-1)+\left(\gamma^{1-\frac{a^2}{t^2}}-1\right)\left(\gamma^{1-\frac{b^2}{t^2}}-1\right)}}.$$

In Theorem 4, if $t = 1$, then the $\mathcal{T}_N^F$ and $\mathcal{T}_C^F$ will reduce to t-norm $T_{F,\varphi}$ and t-conorm $S_{F,\varphi}$ under the Pythagorean fuzzy environment

$$T_{F,\varphi}(a,b) = \sqrt{log_\gamma\frac{(\gamma-1)+\left(\gamma^{a^2}-1\right)\left(\gamma^{b^2}-1\right)}{\gamma-1}},$$
$$S_{F,\varphi}(a,b) = \sqrt{log_\gamma\frac{\gamma(\gamma-1)}{(\gamma-1)+\left(\gamma^{1-a^2}-1\right)\left(\gamma^{1-b^2}-1\right)}}.$$

- Let generator of the linguistic Pythagorean Dombi TN $\mathcal{T}_N^D$ be $\xi(a) = \left(\frac{t^2}{a^2}-1\right)^\gamma$, where $\xi^{-1}(a) = \frac{t}{\sqrt{1+xa^{\frac{1}{\gamma}}}}$ and the generator of the linguistic Pythagorean Dombi TC $\mathcal{T}_C^D$ be $\zeta(a) = \left(\frac{t^2}{a^2}-1\right)^{-\gamma}$, where $\xi^{-1}(a) = \frac{t}{\sqrt{1+a^{-\frac{1}{\gamma}}}}$, where $\gamma \geq 1$.

According to the generators of $\mathcal{T}_N^D$ and $\mathcal{T}_C^D$, similar to proof of Theorem 2, we have

$$\mathcal{T}_N^H(a,b)=\frac{\sqrt{\gamma}tab}{\sqrt{(\gamma-1)a^2b^2+(\gamma t^2-(1-\gamma)a^2)(\gamma t^2-(1-\gamma)a^2)}}$$
$$\mathcal{T}_C^H(a,b)=t\sqrt{\frac{t^2(a^2+b^2)+(\gamma-2)a^2b^2}{t^4+(\gamma-1)a^2b^2}}.$$

Theorem 5: For any $a,b\in[0,t]$, the $\mathcal{T}_N^D$ and $\mathcal{T}_C^D$ can be described by

$$\mathcal{T}_N^D(a,b)=\frac{t}{\sqrt{1+\left(\left(\frac{t^2}{a^2}-1\right)^{\gamma}+\left(\frac{t^2}{b^2}-1\right)^{\gamma}\right)^{\frac{1}{\gamma}}}},$$
$$\mathcal{T}_C^D(a,b)=\frac{t}{\sqrt{1+\left(\left(\frac{t^2}{a^2}-1\right)^{-\gamma}+\left(\frac{t^2}{b^2}-1\right)^{-\gamma}\right)^{-\frac{1}{\gamma}}}}.$$

In Theorem 5, if $t=1$, then the $\mathcal{T}_N^D$ and $\mathcal{T}_C^D$ will reduce to t-norm $T_{D,\gamma}$ and t-conorm $S_{D,\gamma}$ under the Pythagorean fuzzy environment

$$T_{D,\gamma}(a,b)=\frac{1}{\sqrt{1+\left(\left(\frac{1}{a^2}-1\right)^{\gamma}+\left(\frac{1}{b^2}-1\right)^{\gamma}\right)^{\frac{1}{\gamma}}}},$$
$$S_{D,\gamma}(a,b)=\frac{1}{\sqrt{1+\left(\left(\frac{1}{a^2}-1\right)^{-\gamma}+\left(\frac{1}{b^2}-1\right)^{-\gamma}\right)^{-\frac{1}{\gamma}}}}.$$

In the following section, if there is no specific, T and S are above mentioned five types LPTNs and LPTCs, ξ and ζ are the above mentioned generators of LPTNs and LPTCs, respectively.

V. LINGUISTIC PYTHAGOREAN FUZZY AVERAGING OPERATORS BASED ON LPTNS AND LPTCS

In this part, we will give the unified form of some linguistic Pythagorean aggregation operators based the LPTNs and LPTCs introduced in Section 4. Before the unified form given, the operational law should be given firstly.

A. OPERATIONAL LAWS BASED ON LPTNS AND LPTCS

Definition 8: Let $A=(s_{u_1},s_{v_1})$ and $B=(s_{u_1},s_{v_1})$ be two LPFNs, the operational laws of LPFN based on LPTN T and LPTC S are defined as follows:

(1) $A\oplus B=\left(s_{S(u_1,u_2)},s_{T(v_1,v_2)}\right)$
$=\left(s_{\zeta^{-1}(\zeta(u_1)+\zeta(u_2))},s_{\xi^{-1}(\xi(v_1)+\xi(v_2))}\right);$
(2) $A\otimes B=\left(s_{T(u_1,u_2)},s_{S(v_1,v_2)}\right)$
$=\left(s_{\xi^{-1}(\xi(u_1)+\xi(u_2))},s_{\zeta^{-1}(\zeta(v_1)+\zeta(v_2))}\right);$
(3) $\lambda A=\left(s_{\zeta^{-1}(\Psi\zeta(u_1))},s_{\xi^{-1}(\lambda\xi(v_1))}\right);$
(4) $A^{\lambda}=\left(s_{\xi^{-1}(\lambda\xi(u_1))},s_{\zeta^{-1}(\lambda\zeta(v_1))}\right).$

According to above definition, we have the following operational law hold.

Theorem 6: Let A,B,C be three LPFNs, $a,b,c\in R$ and $a,b,c>0$, then we have

(1) $A\oplus B=B\oplus A;$
(2) $(A\oplus B)\oplus C=A\oplus(B\oplus C);$
(3) $aA\oplus bA=(a+b)A;$
(4) $c(aA\oplus bB)=acA\oplus bcB;$
(5) $a(bA)=(ab)A;$
(6) $A\otimes B=B\otimes A;$
(7) $(A\otimes B)\otimes C=A\otimes(B\otimes C).$

Theorem 7: Let $A=(s_{u_1},s_{v_1})$ and $B=(s_{u_2},s_{v_2})$ be two LPFNs, then for any $\lambda>0$, $A\oplus B$, $A\otimes B$, ΨA, A^{λ} are all LPFNs.

Proof. For convenience, the LMD and LNMD of $A\otimes B$, λA, A^{λ} are denoted as s_u,s_v, respectively. In order to prove $A\otimes B$, λA, A^{λ} are LPFNs, the following two aspects must be proven:

(1) $s_0\le s_u,s_v\le s_t$;
(2) $0\le u^2+v^2\le t^2$.

Now, firstly, we prove $A\oplus B$ satisfies the above two conditions. Assume that $A\oplus B=(s_u,s_v)=\left(s_{S(u_1,u_2)},s_{T(v_1,v_2)}\right)=\left(s_{\zeta^{-1}(\zeta(u_1)+\zeta(u_2))},s_{\xi^{-1}(\xi(v_1)+\xi(v_2))}\right).$

As $A=(s_{u_1},s_{v_1})$ and $B=(s_{u_2},s_{v_2})$ are LPFNs, it follows that $s_0\le s_{u_1},s_{u_2}\le s_t$. Since $\zeta(x),\zeta^{-1}(x)$ are monotonically increasing function, $u=S(u_1,u_2)=\zeta^{-1}(\zeta(u_1)+\zeta(u_2))$ and $u_1,u_2\in[0,t]$, it follows that $\zeta(u_1),\zeta(u_2)\in\mathbf{R}^+$ and $\zeta^{-1}(\zeta(u_1)+\zeta(u_2))\in[0,t]$, that is, $s_0\le s_u\le s_t$. Similarly, we can prove $s_0\le s_v\le s_t$. Therefore, s_u,s_v satisfy condition (1).

Because $A=(s_{u_1},s_{v_1})$ and $B=(s_{u_2},s_{v_2})$ are LPFNs, so we have $0\le u_1^2+v_1^2\le t^2$, $0\le u_2^2+v_2^2\le t^2$. According the definitions, we have

$$u^2+v^2=(S(u_1,u_2))^2+(T(v_1,v_2))^2$$
$$=\left(\zeta^{-1}(\zeta(u_1)+\zeta(u_2))\right)^2+\left(\xi^{-1}(\xi(v_1)+\xi(v_2))\right)^2$$

Because $\zeta(a)=\xi\left(\sqrt{t^2-a^2}\right)$, and so $\left(\zeta^{-1}(a)\right)^2=t^2-\left(\xi^{-1}(a)\right)^2$, thus

$$\begin{aligned}u^2&=\left(\zeta^{-1}(\zeta(u_1)+\zeta(u_2))\right)^2\\&=t^2-\left(\xi^{-1}(\zeta(u_1)+\zeta(u_2))\right)^2\\&=t^2-\left(\xi^{-1}\left(\xi\left(\sqrt{t^2-u_1^2}\right)+\xi\left(\sqrt{t^2-u_2^2}\right)\right)\right)^2\end{aligned}$$

As $u_1^2+v_1^2\le t^2,u_2^2+v_2^2\le t^2$, so $v_1\le\sqrt{t^2-u_1^2},v_2\le\sqrt{t^2-u_2^2}$. Because $\xi(x)$ and $\xi^{-1}(x)$ are monotonically decreasing functions, so we have $\xi(v_1) \geq \xi\left(\sqrt{t^2-u_1^2}\right), \xi(v_2) \geq \xi\left(\sqrt{t^2-u_2^2}\right)$. Therefore,

$$\xi(v_1)+\xi(v_2) \geq \xi\left(\sqrt{t^2-u_1^2}\right)+\xi\left(\sqrt{t^2-u_2^2}\right).$$

Furthermore,

$$\xi^{-1}(\xi(v_1)+\xi(v_2)) \leq \xi^{-1}\left(\xi\left(\sqrt{t^2-u_1^2}\right)+\xi\left(\sqrt{t^2-u_2^2}\right)\right).$$

and

$$\left(\xi^{-1}(\xi(v_1)+\xi(v_2))\right)^2 \leq \left(\xi^{-1}\left(\xi\left(\sqrt{t^2-u_1^2}\right)+\xi\left(\sqrt{t^2-u_2^2}\right)\right)\right)^2.$$

That is,

$$\begin{aligned} v^2 &\leq \left(\xi^{-1}\left(\xi\left(\sqrt{t^2-u_1^2}\right)+\xi\left(\sqrt{t^2-u_2^2}\right)\right)\right)^2 \\ &= t^2-u^2. \end{aligned}$$

Therefore, $u^2+v^2 \leq t^2$. That is, $A \oplus B$ is a LPFN. Similarly, we can prove $A \otimes B, \lambda A, A^{\lambda}$ are LPFNs, so the details are omitted.

B. LINGUISTIC PYTHAGOREAN FUZZY AVERAGING OPERATORS BASED ON LPTNS AND LPTCS

In this subpart, the Linguistic Pythagorean fuzzy averaging operators based on LPTNs and LPTCs will be given.

Definition 9: Let $\mathbb{A} = \{A_i = (s_{u_i}, s_{v_i}) | i = 1, 2, \cdots, n\}$ be a collection of LPFNs and ω_i be the weight of A_i with $w_i > 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$. The linguistic Pythagorean fuzzy weighted averaging operator (LPFWA) is defined as follows:

$$LPFWA(A_1, A_2, \cdots, A_n) = \omega_1 A_1 \oplus \omega_2 A_2 \oplus \cdots \oplus \omega_n A_n.$$

Theorem 8: Let $\mathbb{A} = \{A_i = (s_{u_i}, s_{v_i}) | i = 1, 2, \cdots, n\}$ be a collection of LPFNs and ω_i be the weight of A_i with $w_i > 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$. Then

$$\begin{aligned} &LPFWA(A_1, A_2, \cdots, A_n) \\ &= \left(s_{\zeta^{-1}\left(\sum_{i=1}^{n}(\omega_i\zeta(u_i))\right)}, s_{\xi^{-1}\left(\sum_{i=1}^{n}(\omega_i\xi(v_i))\right)}\right). \end{aligned} \quad (5)$$

Proof. Theorem 8 can be proved by the mathematical induction method.

(1) When $n = 1$, Theorem 8 is held.

(2) Assume that theorem 8 is held when $n = k$, that is,

$$LPFWA(A_1, A_2, \cdots, A_k) = \left(s_{\zeta^{-1}\left(\sum_{i=1}^{k}(\omega_i\zeta(u_i))\right)}, s_{\xi^{-1}\left(\sum_{i=1}^{k}(\omega_i\xi(v_i))\right)}\right).$$

Then, when $n = k+1$, it follows from induction hypothesis that

$$\begin{aligned} &LPFWA(A_1, A_2, \cdots, A_{k+1}) \\ &= \left(s_{\zeta^{-1}\left(\sum_{i=1}^{k}(\omega_i\zeta(u_i))\right)}, s_{\xi^{-1}\left(\sum_{i=1}^{k}(\omega_i\xi(v_i))\right)}\right) \\ &\oplus \left(s_{\zeta^{-1}(\omega_{k+1}\zeta(u_{k+1}))}, s_{\xi^{-1}(\omega_{k+1}\xi(v_{k+1}))}\right) \\ &= \left(s_{\zeta^{-1}\left(\zeta\left(\zeta^{-1}\left(\sum_{i=1}^{k}(\omega_i\zeta(u_i))\right)\right)+\zeta\left(\zeta^{-1}(\omega_{k+1}\zeta(u_{k+1}))\right)\right)},\right. \\ &\left. s_{\xi^{-1}\left(\xi\left(\xi^{-1}\left(\sum_{i=1}^{k}(\omega_i\xi(v_i))\right)\right)+\xi\left(\xi^{-1}(\omega_{k+1}\xi(v_{k+1}))\right)\right)}\right) \\ &= \left(s_{\zeta^{-1}\left(\sum_{i=1}^{k+1}(\omega_i\zeta(u_i))\right)}, s_{\xi^{-1}\left(\sum_{i=1}^{k+1}(\omega_i\xi(v_i))\right)}\right). \end{aligned}$$

Thus, theorem 8 holds for all positive integer n.

Theorem 9: Let $\mathbb{A} = \{A_i = (s_{u_i}, s_{v_i}) | i = 1, 2, \cdots, n\}$ be a collection of LPFNs and ω_i be the weight of A_i with $w_i > 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$. The the following properties hold:

(1)(**Idempotency**) If $A_i = A = (s_u, s_v)$ for $i = 1, 2, \cdots, n$, then

$$LPFWA(A_1, A_2, \cdots, A_n) = A = (s_u, s_v).$$

(2) (**Monotonicity**) Let $\mathbb{B} = \left\{B_i = \left(s_{u_i^{'}}, s_{v_i^{'}}\right) | i = 1, 2, \cdots, n\right\}$ be another collection of LPFNs such that $s_{u_i} \leq s_{u_i^{'}}$ and $s_{v_i} \geq s_{v_i^{'}}$ for all i, then

$$LPFWA(A_1, A_2, \cdots, A_n) \leq LPFWA(B_1, B_2, \cdots, B_n).$$

(3)(**Boudedness**)

$$\begin{aligned} (min_i(s_{u_i}), max_i(s_{v_i})) &\leq LPFWA(A_1, A_2, \cdots, A_n) \\ &\leq (max_i(s_{u_i}), min_i(s_{v_i})). \end{aligned}$$

Proof. (1) If $A_i = A = (s_u, s_v)$ for $i = 1, 2, \cdots, n$, then

$$\begin{aligned} &LPFWA(A_1, A_2, \cdots, A_n) \\ &= \left(s_{\zeta^{-1}\left(\sum_{i=1}^{n}(\omega_i\zeta(u_i))\right)}, s_{\xi^{-1}\left(\sum_{i=1}^{n}(\omega_i\xi(v_i))\right)}\right) \\ &= \left(s_{\zeta^{-1}\left(\sum_{i=1}^{n}(\omega_i\zeta(u))\right)}, s_{\xi^{-1}\left(\sum_{i=1}^{n}(\omega_i\xi(v))\right)}\right) \\ &= \left(s_{\zeta^{-1}(\zeta(u))}, s_{\xi^{-1}(\xi(v))}\right) \\ &= \left(s_{\zeta^{-1}(\zeta(u))}, s_{\xi^{-1}(\xi(v))}\right) \\ &= (s_u, s_v). \end{aligned}$$

(2) since $s_{u_i} \leq s_{u_i^{'}}$ and $s_{v_i} \geq s_{v_i^{'}}$, it follows that $u_i \leq u_i^{'}$ and $v_i \geq v_i^{'}$. Because $\zeta(x)$ and $\zeta^{-1}(x)$ are monotonicity increasing function, we have $\zeta(u_i) \leq \zeta\left(u_i^{'}\right)$, furthermore, $\sum_i^n (\omega_i\zeta(u_i)) \leq \sum_i^n \left(\omega_i\zeta\left(u_i^{'}\right)\right)$. And so $\zeta^{-1}\left(\sum_i^n(\omega_i\zeta(u_i))\right) \leq \zeta^{-1}\left(\sum_i^n\left(\omega_i\zeta\left(u_i^{'}\right)\right)\right)$.

Since $\xi(x)$ and $\xi^{-1}(x)$ are monotonicity decreasing function, we have $\xi(v_i) \geq \xi\left(v_i^{'}\right)$, furthermore, $\sum_i^n(\omega_i\xi(v_i)) \geq \sum_i^n\left(\omega_i\xi\left(v_i^{'}\right)\right)$. And so $\xi^{-1}\left(\sum_i^n(\omega_i\xi(v_i))\right) \leq \xi^{-1}\left(\sum_i^n\left(\omega_i\xi\left(v_i^{'}\right)\right)\right)$. Therefore,

$$\begin{aligned} &\left(s_{\zeta^{-1}\left(\sum_i^n(\omega_i\zeta(u_i))\right)}, s_{\xi^{-1}\left(\sum_i^n(\omega_i\xi(v_i))\right)}\right) \\ &\leq \left(s_{\zeta^{-1}\left(\sum_i^n\left(\omega_i\zeta(u_i^{'})\right)\right)}, s_{\xi^{-1}\left(\sum_i^n\left(\omega_i\xi(v_i^{'})\right)\right)}\right), \end{aligned}$$

that is,

$$LPFWA(A_1, A_2, \cdots, A_n) \leq LPFWA(B_1, B_2, \cdots, B_n).$$

(3)This property is obvious from idempotency and monotonicity, so the detail of the proof is omitted.

C. GENERALIZED LINGUISTIC PYTHAGOREAN FUZZY AVERAGING OPERATORS BASED ON LPTNS AND LPTCS

In this subpart, we will introduce the generalized linguistic Pythagorean fuzzy weighted averaging operators (GLPFWA) based on the LPTNs and LPTCs.

Definition 10: Let $\mathbb{A} = \{A_i = (s_{u_i}, s_{v_i}) | i = 1, 2, \cdots, n\}$ be a collection of LPFNs and ω_i be the weight of A_i with $w_i > 0\ (i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$. The GLPFWA is defined as follows:

$$GLPFWA(A_1, A_2, \cdots, A_n) = \left(\omega_1 A_1^\lambda \oplus \omega_2 A_2^\lambda \oplus \cdots \oplus \omega_n A_n^\lambda\right)^{\frac{1}{\lambda}}. \quad (6)$$

where $\lambda > 0$.

Theorem 10: Let $\mathbb{A} = \{A_i = (s_{u_i}, s_{v_i}) | i = 1, 2, \cdots, n\}$ be a collection of LPFNs and ω_i be the weight of A_i with $w_i > 0\ (i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$. Then

$$GLPFWA(A_1, A_2, \cdots, A_n) = (s_u, s_v),$$

where

$$u = \xi^{-1}\left(\frac{1}{\lambda}\xi\left(\zeta^{-1}\left(\sum_{i=1}^{n}\left(\omega_i\zeta\left(\xi^{-1}\left(\lambda\xi(u_i)\right)\right)\right)\right)\right)\right),$$
$$v = \zeta^{-1}\left(\frac{1}{\lambda}\zeta\left(\xi^{-1}\left(\sum_{i=1}^{n}\left(\omega_i\xi\left(\zeta^{-1}\left(\lambda\zeta(v_i)\right)\right)\right)\right)\right)\right).$$

Proof. Since $A_i^\lambda = \left(s_{\xi^{-1}(\lambda\xi(u_i))}, s_{\zeta^{-1}(\lambda\zeta(v_i))}\right)$, it follows from Theorem 8 that

$$\omega_1 A_1^\lambda \oplus \omega_2 A_2^\lambda \oplus \cdots \oplus \omega_n A_n^\lambda = (s_u, s_v),$$

where

$$u = \zeta^{-1}\left(\sum_{i=1}^{n}\left(\omega_i\zeta\left(\xi^{-1}\left(\lambda\xi(u_i)\right)\right)\right)\right),$$
$$v = \xi^{-1}\left(\sum_{i=1}^{n}\left(\omega_i\xi\left(\zeta^{-1}\left(\lambda\zeta(v_i)\right)\right)\right)\right).$$

Therefore,

$$\left(\omega_1 A_1^\lambda \oplus \omega_2 A_2^\lambda \oplus \cdots \oplus \omega_n A_n^\lambda\right)^{\frac{1}{\lambda}} = (s_u, s_v),$$

where

$$u = \xi^{-1}\left(\frac{1}{\lambda}\xi\left(\zeta^{-1}\left(\sum_{i=1}^{n}\left(\omega_i\zeta\left(\xi^{-1}\left(\lambda\xi(u_i)\right)\right)\right)\right)\right)\right),$$
$$v = \zeta^{-1}\left(\frac{1}{\lambda}\zeta\left(\xi^{-1}\left(\sum_{i=1}^{n}\left(\omega_i\xi\left(\zeta^{-1}\left(\lambda\zeta(v_i)\right)\right)\right)\right)\right)\right)$$

That is,

$$GLPFWA(A_1, A_2, \cdots, A_n) = (s_u, s_v),$$

where

$$u = \xi^{-1}\left(\frac{1}{\lambda}\xi\left(\zeta^{-1}\left(\sum_{i=1}^{n}\left(\omega_i\zeta\left(\xi^{-1}\left(\lambda\xi(u_i)\right)\right)\right)\right)\right)\right),$$
$$v = \zeta^{-1}\left(\frac{1}{\lambda}\zeta\left(\xi^{-1}\left(\sum_{i=1}^{n}\left(\omega_i\xi\left(\zeta^{-1}\left(\lambda\zeta(v_i)\right)\right)\right)\right)\right)\right)$$

Similarly, we give some properties of GLPFWA, the details of the proofs of these properties are omitted.

Theorem 11: Let $\mathbb{A} = \{A_i = (s_{u_i}, s_{v_i}) | i = 1, 2, \cdots, n\}$ be a collection of LPFNs and ω_i be the weight of A_i with $w_i > 0\ (i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$. The the following properties hold:

(1) (**Idempotency**) If $A_i = A = (s_u, s_v)$ for $i = 1, 2, \cdots, n$, then

$$GLPFWA(A_1, A_2, \cdots, A_n) = A = (s_u, s_v).$$

(2) (**Monotonicity**) Let $\mathbb{B} = \{B_i = \left(s_{u_i'}, s_{v_i'}\right) | i = 1, 2, \cdots, n\}$ be another collection of LPFNs such that $s_{u_i} \leq s_{u_i'}$ and $s_{v_i} \geq s_{v_i'}$ for all i, then

$$GLPFWA(A_1, A_2, \cdots, A_n) \leq GLPFWA(B_1, B_2, \cdots, B_n).$$

(3)(**Boudedness**)

$$(min_i(s_{u_i}), max_i(s_{v_i})) \leq GLPFWA(A_1, A_2, \cdots, A_n) \leq (max_i(s_{u_i}), min_i(s_{v_i})).$$

D. ANALYSES OF GLPFWA

In this subpart, we consider different types of GLPFWA by considering parameter λ and different types of LPTNs and LPTCs.

1) Analyzes of Parameter λ

- When $\lambda = 1$, GLPFWA will reduce to the following:

$$LPFWA(A_1, A_2, \cdots, A_n) = \left(s_{\zeta^{-1}\left(\sum_{i=1}^{n}(\omega_i\zeta(u_i))\right)}, s_{\xi^{-1}\left(\sum_{i=1}^{n}(\omega_i\xi(v_i))\right)}\right),$$

which is called linguistic Pythagorean fuzzy weighted averaging (LPFWA) operator based on LPTNs and LPTCs. In this situation, if $w = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)$, then LPFWA operator will reduce to the following:

$$LPFA(A_1, A_2, \cdots, A_n) = \left(s_{\zeta^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\zeta(u_i)\right)}, s_{\xi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\xi(v_i)\right)}\right),$$

which is called linguistic Pythagorean fuzzy averaging (LPFA) operator.

- When $\lambda = 2$, GLPFWA will reduce to linguistic Pythagorean fuzzy quadratic averaging aggregation (LPFQWA) operator

$$LPFWA(A_1, A_2, \cdots, A_n) = (s_u, s_v).$$

where

$$u=\xi^{-1}\left(\frac{1}{2}\xi\left(\zeta^{-1}\left(\sum_{i=1}^{n}\left(\omega_i\zeta\left(\xi^{-1}\left(2\xi\left(u_i\right)\right)\right)\right)\right)\right)\right),$$
$$v=\zeta^{-1}\left(\frac{1}{2}\zeta\left(\xi^{-1}\left(\sum_{i=1}^{n}\left(\omega_i\xi\left(\zeta^{-1}\left(2\zeta\left(v_i\right)\right)\right)\right)\right)\right)\right).$$

- When $\lambda\to 0$, GLPFWA will reduce to the following:

$$LPFWG\left(A_1,A_2,\cdots,A_n\right)=\left(s_{\xi^{-1}\left(\sum_{i=1}^{n}\left(\omega_i\xi\left(u_i\right)\right)\right)},s_{\zeta^{-1}\left(\sum_{i=1}^{n}\left(\omega_i\zeta\left(v_i\right)\right)\right)}\right),$$

which is called linguistic Pythagorean fuzzy weighted geometric (LPFWG)operator based on LPTNs and LPTCs. In Eq.(22), if $w=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$ based on LPTNs and LPTCs, then LPFWG operator will reduce to the following:

$$LPFG\left(A_1,A_2,\cdots,A_n\right)=\left(s_{\xi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\xi\left(u_i\right)\right)},s_{\zeta^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\zeta\left(v_i\right)\right)}\right),$$

which is called linguistic Pythagorean fuzzy geometric (LPFG) operator.

2) Analyzes of PLFWA operators based on different LPTNs and LPTCs

We will obtain some different aggregation operators by different LPTNs and extended LPTCs given in Section 3.

Case I: When $T=\mathcal{T}_N^A$ and $S=\mathcal{T}_C^A$, that is, $\xi\left(a\right)=-ln\left(\frac{a}{t}\right)^2$ and $\zeta\left(a\right)=-ln\frac{t^2-a^2}{t^2}$, then the GLPFWA will be generalized linguistic Pythagorean fuzzy weighted averaging aggregation (GLPFWA) operator:

$$GLPFWA\left(A_1,A_2,\cdots,A_n\right)=\left(s_u,s_v\right). \tag{7}$$

where

$$u=t\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\left(\frac{u_i^2}{t^2}\right)^{\lambda}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}},$$
$$v=t\sqrt{1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\frac{v_i^2}{t^2}\right)^{\lambda}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}}.$$

In this case, if $\lambda=1$, the GLPFWA will be linguistic Pythagorean fuzzy weighted averaging aggregation operator [40]

$$LPFWA\left(A_1,A_2,\cdots,A_n\right)=\left(s_{\sqrt{t^2-t^2\prod_{i=1}^{n}\left(\frac{t^2-u_i^2}{t^2}\right)^{\omega_i}}},s_{t\prod_{i=1}^{n}\left(\frac{v_i}{t}\right)^{\omega_i}}\right).$$

when $\lambda\to 0$, the GLPFWA will be linguistic Pythagorean fuzzy weighted geometric (LPFGA) operator [40]

$$LPFGA\left(A_1,A_2,\cdots,A_n\right)=\left(s_{t\prod_{i=1}^{n}\left(\frac{v_i}{t}\right)^{\omega_i}},s_{\sqrt{t^2-t^2\prod_{i=1}^{n}\left(\frac{t^2-u_i^2}{t^2}\right)^{\omega_i}}}\right).$$

Case II: When $T=\mathcal{T}_N^E$ and $S=\mathcal{T}_C^E$, that is, $\xi\left(a\right)=ln\left(\frac{2t^2-a^2}{a^2}\right)$ and $\zeta\left(a\right)=ln\frac{t^2+a^2}{t^2-a^2}$, the GLPFWA will be generalized linguistic Pythagorean fuzzy Einstein averaging aggregation (GELPFEWA) operator

$$GLPFEWA\left(A_1,A_2,\cdots,A_n\right)=\left(s_{\frac{\sqrt{2}t}{\sqrt{1+\left(\frac{a+3b}{a-b}\right)^{\frac{1}{\lambda}}}}},s_{t\sqrt{\frac{(c+3d)^{\frac{1}{\lambda}}-(c-d)^{\frac{1}{\lambda}}}{(c+3d)^{\frac{1}{\lambda}}+(c-d)^{\frac{1}{\lambda}}}}}\right), \tag{8}$$

where

$$a=\prod_{i=1}^{n}\left(\left(2t^2-u_i^2\right)^{\lambda}+3\left(u_i^2\right)^{\lambda}\right)^{\omega_i},$$
$$b=\prod_{i=1}^{n}\left(\left(2t^2-u_i^2\right)^{\lambda}-\left(u_i^2\right)^{\lambda}\right)^{\omega_i},$$
$$c=\prod_{i=1}^{n}\left(\left(t^2+v_i^2\right)^{\lambda}+3\left(t^2-v_i^2\right)^{\lambda}\right)^{\omega_i},$$
$$d=\prod_{i=1}^{n}\left(\left(t^2+v_i^2\right)^{\lambda}-\left(t^2-v_i^2\right)^{\lambda}\right)^{\omega_i}.$$

In this case, when $\lambda=1$, the GLPFWA will be linguistic Pythagorean fuzzy Einstein weighted averaging (PLFEWA) operator

$$LPFEWA\left(A_1,A_2,\cdots,A_n\right)=\left(s_u,s_v\right).$$

where

$$u=t\sqrt{\frac{\prod_{i=1}^{n}\left(t^2+u_i^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(t^2-u_i^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(t^2+u_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(t^2-u_i^2\right)^{\omega_i}}},$$
$$v=\frac{\sqrt{2}t\prod_{i=1}^{n}v_i^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(v_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(2t^2-v_i^2\right)^{\omega_i}}}.$$

When $\lambda\to 0$, the GLPFWA will be linguistic Pythagorean fuzzy Einstein weighted geometric (LPFEWG) operator

$$LPFEWG\left(A_1,A_2,\cdots,A_n\right)=\left(s_u,s_v\right) \tag{9}$$

where

$$u=\frac{\sqrt{2}t\prod_{i=1}^{n}v_i^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(v_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(2t^2-v_i^2\right)^{\omega_i}}},$$
$$v=t\sqrt{\frac{\prod_{i=1}^{n}\left(t^2+u_i^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(t^2-u_i^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(t^2+u_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(t^2-u_i^2\right)^{\omega_i}}}.$$

Case III: When $T=\mathcal{T}_N^H$ and $S=\mathcal{T}_C^H$, that is, $\xi\left(a\right)=ln\left(\frac{\gamma t^2+(1-\gamma)a^2}{x^2}\right)$ and $\zeta\left(a\right)=ln\frac{\gamma t^2+(1-\gamma)\left(t^2-a^2\right)}{t^2-a^2}$, the GLPFWA operator will be generalized linguistic Pythagorean fuzzy Hamacher averaging aggregation operator (GLPFHWA)

$$GLPFHWA\left(A_1,A_2,\cdots,A_n\right)=\left(s_u,s_v\right), \tag{10}$$

where

$$u = t\frac{\sqrt{\gamma\left(a-b\right)^{\frac{1}{\lambda}}}}{\sqrt{\left(a-b\right)^{\frac{1}{\lambda}}+\left(a+\left(\gamma^2-1\right)b\right)^{\frac{1}{\lambda}}}},$$

$$v = t\sqrt{\frac{\left(c+\left(\gamma^2-1\right)d\right)^{\frac{1}{\lambda}}-\left(c-d\right)^{\frac{1}{\lambda}}}{\left(c+\left(\gamma^2-1\right)d\right)^{\frac{1}{\lambda}}-\left(\gamma-1\right)\left(c-d\right)^{\frac{1}{\lambda}}}}$$

and

$$a = \prod_{i=1}^{n}\left(\left(\gamma t^2+\left(1-\gamma\right)u_i^2\right)^{\Psi}+\left(\gamma^2-1\right)\left(u_i^2\right)^{\lambda}\right)^{\omega_i},$$

$$b = \prod_{i=1}^{n}\left(\left(\gamma t^2+\left(1-\gamma\right)u_i^2\right)^{\lambda}-\left(u_i^2\right)^{\lambda}\right)^{\omega_i},$$

$$c = \prod_{i=1}^{n}\left(\left(t^2+\left(\gamma-1\right)v_i^2\right)^{\lambda}+\left(\gamma^2-1\right)\left(t^2-v_i^2\right)^{\lambda}\right)^{\omega_i},$$

$$d = \prod_{i=1}^{n}\left(\left(t^2+\left(\gamma-1\right)v_i^2\right)^{\lambda}-\left(t^2-v_i^2\right)^{\lambda}\right)^{\omega_i}.$$

In this case, when $\lambda = 1$, the GLPFHWA will be linguistic Pythagorean fuzzy Hamacher weighted averaging (LPFHWA) operator

$$LPFHWA\left(A_1, A_2, \cdots, A_n\right) = \left(s_u, s_v\right), \tag{11}$$

where

$$u = t\sqrt{\frac{\prod_{i=1}^{n}\left(\gamma t^2+\left(1-\gamma\right)\left(t^2-u_i^2\right)\right)^{\omega_i}-\prod_{i=1}^{n}\left(t^2-u_i^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(\gamma t^2+\left(1-\gamma\right)\left(t^2-u_i^2\right)\right)^{\omega_i}+\left(\gamma-1\right)\prod_{i=1}^{n}\left(t^2-u_i^2\right)^{\omega_i}}}$$

$$v = \frac{\sqrt{\gamma}t\prod_{i=1}^{n}v_i^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(v_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(\gamma t^2+\left(1-\gamma\right)v_i^2\right)^{\omega_i}}}.$$

when $\lambda \to 0$, the GLPFHWA will be linguistic Pythagorean fuzzy Hamacher weighted geometric (LPFHWG) operator

$$LPFHWG\left(A_1, A_2, \cdots, A_n\right) = \left(s_u, s_v\right), \tag{12}$$

where

$$u = \frac{\sqrt{\gamma}t\prod_{i=1}^{n}v_i^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(v_i^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(\gamma t^2+\left(1-\gamma\right)v_i^2\right)^{\omega_i}}},$$

$$v = t\sqrt{\frac{\prod_{i=1}^{n}\left(\gamma t^2+\left(1-\gamma\right)\left(t^2-u_i^2\right)\right)^{\omega_i}-\prod_{i=1}^{n}\left(t^2-u_i^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(\gamma t^2+\left(1-\gamma\right)\left(t^2-u_i^2\right)\right)^{\omega_i}+\left(\gamma-1\right)\prod_{i=1}^{n}\left(t^2-u_i^2\right)^{\omega_i}}}.$$

Case IV: When $T = \mathcal{T}_N^F$ and $S = \mathcal{T}_C^F$, that is, $\xi\left(a\right) = ln\left(\frac{\gamma-1}{\gamma^{\frac{a^2}{t^2}}-1}\right)$ and $\zeta\left(a\right) = ln\left(\frac{\gamma-1}{\gamma^{1-\frac{a^2}{t^2}}-1}\right)$, the GLPFWA will be generalized linguistic Pythagorean fuzzy Frank averaging aggregation (GLPFFWA) operator

$$GLPFFWA\left(A_1, A_2, \cdots, A_n\right) = \left(s_u, s_v\right) \tag{13}$$

where

$$u = t\sqrt{log_{\gamma}\left(1+\left(\gamma-1\right)\left(\frac{\prod_{i=1}^{n}a^{\omega_i}-1}{\prod_{i=1}^{n}a^{\omega_i}+\left(\gamma-1\right)}\right)^{\frac{1}{\lambda}}\right)},$$

$$v = t\sqrt{log_{\gamma}\frac{\gamma\left(\prod_{i=1}^{n}b^{\omega_i}+\left(\gamma-1\right)\right)^{\frac{1}{\lambda}}}{\left(\prod_{i=1}^{n}b^{\omega_i}+\left(\gamma-1\right)\right)^{\frac{1}{\lambda}}+\left(\gamma-1\right)\left(\prod_{i=1}^{n}b^{\omega_i}-1\right)^{\frac{1}{\lambda}}}}.$$

and

$$a = \frac{\left(\gamma-1\right)^{\lambda}+\left(\gamma-1\right)\left(\gamma^{\frac{u_i^2}{t^2}}-1\right)^{\lambda}}{\left(\gamma-1\right)^{\lambda}-\left(\gamma^{\frac{u_i^2}{t^2}}-1\right)^{\lambda}},$$

$$b = \frac{\left(\gamma-1\right)^{\lambda}+\left(\gamma-1\right)\left(\gamma^{1-\frac{u_i^2}{t^2}}-1\right)^{\lambda}}{\left(\gamma-1\right)^{\lambda}-\left(\gamma^{1-\frac{u_i^2}{t^2}}-1\right)^{\lambda}}.$$

In this case, when $\lambda = 1$, the GLPFFWA will be linguistic Pythagorean fuzzy Frank weighted averaging (LPFFWA) operator

$$LPFFWA\left(A_1, A_2, \cdots, A_n\right) = \left(s_u, s_v\right), \tag{14}$$

where

$$u = t\sqrt{1-log_{\gamma}\left(1+\prod_{i=1}^{n}\left(\gamma^{1-\frac{u_i^2}{t^2}}-1\right)^{\omega_i}\right)},$$

$$v = t\sqrt{log_{\gamma}\left(1+\prod_{i=1}^{n}\left(\gamma^{\frac{v_i^2}{t^2}}-1\right)^{\omega_i}\right)}.$$

when $\lambda \to 0$, the GLPFFWA will be linguistic Pythagorean fuzzy Frank weighted geometric (LPFFWG) operator

$$LPFFWG\left(A_1, A_2, \cdots, A_n\right) = \left(s_u, s_v\right), \tag{15}$$

where

$$u = t\sqrt{log_{\gamma}\left(1+\prod_{i=1}^{n}\left(\gamma^{\frac{v_i^2}{t^2}}-1\right)\right)^{\omega_i}},$$

$$v = t\sqrt{1-log_{\gamma}\left(1+\prod_{i=1}^{n}\left(\gamma^{1-\frac{u_i^2}{t^2}}-1\right)\right)^{\omega_i}}.$$

Here, we can prove that when $\gamma \to 1$, LPFFWA will reduce to the linguistic Pythagorean fuzzy weighted averaging aggregation operator [40]

$$LPFWA\left(A_1, A_2, \cdots, A_n\right) = \left(s_u, s_v\right). \tag{16}$$

where

$$u = \sqrt{t^2-t^2\prod_{i=1}^{n}\left(\frac{t^2-u_i^2}{t^2}\right)^{\omega_i}}, v = t\prod_{i=1}^{n}\left(\frac{v_i}{t}\right)^{\omega_i}.$$

Case V: When $T = \mathcal{T}_N^D$ and $S = \mathcal{T}_C^D$, that is, $\xi\left(a\right) = \left(\frac{t^2}{a^2}-1\right)^{\frac{1}{\gamma}}$ and $\zeta\left(a\right) = \left(\frac{t^2}{a^2}-1\right)^{-\gamma}$, the GLPFWA will be generalized linguistic Pythagorean fuzzy Dombi averaging aggregation (GLPFDWA) operator

$$GLPFDWA\left(A_1, A_2, \cdots, A_n\right) = \left(s_u, s_v\right), \tag{17}$$

where

$$u = t\sqrt{1+\left(\sum_{i=1}^{n} w_i\left(\frac{t^2}{u_i^2}-1\right)^{-\gamma}\right)^{-\frac{1}{\gamma}}},$$

$$v = t\sqrt{1+\left(\sum_{i=1}^{n} w_i\left(\frac{t^2}{u_i^2}-1\right)^{\gamma}\right)^{\frac{1}{\gamma}}}.$$

Remark 5. GLPFWA provides a parameterized family for linguistic Pythagorean fuzzy aggregation operators. According to different LPTNs, LPTCs and parameter Ψ, we can obtain a wide range of linguistic Pythagorean fuzzy aggregation operators, such as GLPFWA, GLPFEWA, GLPFHWA, GLPFFWA, GLPFDWA. The main advantage of these operators is that it includes a wide range of specific cases that enables us to consider many different decision making situations and select the one that best fits with our interests.

VI. APPROACH FOR LINGUISTIC PYTHAGOREAN MAGDM

In this part, we will give a approach for linguistic Pythagorean MAGDM (LPMAGDM) with unknown expert weights and attribute weights. To do so, the linguistic Pythagorean cross-entropy is first introduced and the approach for fixing the expert weights based on the proposed linguistic Pythagorean cross-entropy is built; Secondly, and also the method for calculating the attribute weight based on the proposed distance measures in Section 3; Finally, the the LPMAGDM method is also established.

A. FORMAL REPRESENTATION OF LINGUISTIC PYTHAGOREAN MAGDM

Generally speaking, MAGDM is always used to find the best one from a finite set of alternatives w. r. t. predefined attributes. LPMAGDM method aims at handling the MAGDM problems with LPFI, especially the MAGDM problems are related to subjective information and attitudinal characteristics of decision makers. A LPMAGDM problem can be formally described as follows:

(1) $\Xi = \{\Psi_1, \Psi_2, \cdots, \Psi_m\}$ a collection of m alternatives;

(2) $A = \{a_1, a_2, \cdots, a_n\}$ a collection of n attributes whose weight vector is $w = (w_1, \cdots, w_n)$ with $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$;

(3)$E = (e_1, e_2, \cdots, e_q)$ a collection of q experts $w = (\lambda_1, \cdots, \lambda_q)$ with $\lambda_i > 0$ and $\sum_{i=1}^{q} \lambda_i = 1$;

(4)The kth decision maker provides the attribute values of alternative $\Psi_i \in \Xi\,(i = 1, 2, \cdots, m)$ w. r. t. attribute $a_j\,(j = 1, 2, \cdots, n)$ and construct the linguistic Pythagorean fuzzy decision making matric

$$R^{(k)} = \left(\alpha_{ij}^{(k)}\right)_{m\times n} = \begin{pmatrix} \left(s_{u_{11}}^{(k)}, s_{v_{11}}^{(k)}\right) & \left(s_{u_{12}}^{(k)}, s_{v_{12}}^{(k)}\right) & \cdots & \left(s_{u_{1n}}^{(k)}, s_{v_{1n}}^{(k)}\right) \\ \left(s_{u_{21}}^{(k)}, s_{v_{21}}^{(k)}\right) & \left(s_{u_{22}}^{(k)}, s_{v_{22}}^{(k)}\right) & \cdots & \left(s_{u_{2n}}^{(k)}, s_{v_{2n}}^{(k)}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \left(s_{u_{m1}}^{(k)}, s_{v_{m1}}^{(k)}\right) & \left(s_{u_{m2}}^{(k)}, s_{v_{m2}}^{(k)}\right) & \cdots & \left(s_{u_{mn}}^{(k)}, s_{v_{mn}}^{(k)}\right) \end{pmatrix}$$

where $s_{u_{ij}}, s_{v_{ij}} \in \bar{\mathfrak{S}}$ is a LPFN, $s_{u_{ij}}$ is the LMD in which alternative Ψ_i should satisfy the attribute a_j, $s_{v_{ij}}$ is the LNMD in which alternative Ψ_i should not satisfy the attribute a_j , and $s_0 \le s_{u_{ij}}, s_{v_{ij}} \le s_t$, $0 \le u_{ij}^2 + v_{ij}^2 \le t^2$.

B. DETERMINATION OF EXPERTS' WEIGHTS BASED ON LINGUISTIC PYTHAGOREAN CROSS-ENTROPY

There is such a decision problems in which the information about attribute weights is incompletely known or completely unknown due to time pressure, lack of knowledge and the expert's limited expertise about the problem domain. Therefore, it is necessary to investigate this issue. There are some approaches [50]–[53], [53], [53]–[55] for MADM (MAGDM) problems with the experts or (and) attributes weights are incompletely known or completely unknown.

To establish the approach for dealing with LPMAGDM problem with unknown experts' weights, so it is necessary to determine the experts' weights before introducing the process of LPMAGDM. To derive the experts' weights, we first define the concept linguistic Pythagorean cross entropy. In real decision problem, the hesitation degree of information should be considered for supporting membership and nonmenbership functions. So DM's preference supports should be considered when giving the definition of cross entropy.

Definition 11: Let $\mathbb{A} = \{(s_{u_i}, s_{v_i})\,|i = 1, 2, \cdots, n\}$ and $\mathbb{B} = \{\left(s_{u_i'}, s_{v_i'}\right)|i = 1, 2, \cdots, n\}$ be two sets of linguistic Pythagorean numbers with $s_{u_i}, s_{v_i}, s_{u_i'}, s_{v_i'} \in \bar{\mathfrak{S}} = \{s_\alpha | s_0 \le s_\alpha \le s_t | \alpha \in [0, t]\}$. The linguistic Pythagorean cross entropy $E_C\,(\mathbb{A}, \mathbb{B})$ of A and B is defined as

where $s_{\pi_i} = s_{\sqrt{t^2-u_i^2-v_i^2}}, s_{\pi_i'} = s_{\sqrt{t^2-(u_i')^2-(v_i')^2}}$ are the linguistic indeterminacy degree and $q > 0$.

It follows easily from Shannon inequality that the following properties hold.

(1) $E_C\,(\mathbb{A}, \mathbb{B}) \ge 0$;

(2) $E_C\,(\mathbb{A}, \mathbb{B}) = 0 \Leftrightarrow \mathbb{A} = \mathbb{B}$ and

(3) $E_C\,(\mathbb{A}, \mathbb{B}) = E_C\,(\mathbb{B}, \mathbb{A})$.

The proposed linguistic Pythagorean cross-entropy is a useful tool to measure the degree of discrimination of individual decision matrices (IDMs) from positive or negative ideal decision matrix. Qi [56] devise relative discrimination measure to represent the credibility of an IDM under IVIF environment. With the help of this thought, we establish a new model based on the linguistic Pythagorean cross-entropy for fixing the experts weights if the experts weights information are unknown completely.

$$E_C(\mathbb{A},\mathbb{B}) = \frac{1}{2t^2}\sum_{i=1}^{n}\Bigg\{\Bigg[\left(t^2+qu_i^2\right)ln\left(\frac{t^2+qu_i^2}{t^2}\right)+\left(t^2+q\left(u_i^{'}\right)^2\right)ln\left(\frac{t^2+q\left(u_i^{'}\right)^2}{t^2}\right)$$
$$-\left(2t^2+q\left(u_i^2+\left(u_i^{'}\right)^2\right)\right)ln\left(\frac{2t^2+q\left(u_i^2+\left(u_i^{'}\right)^2\right)}{2t^2}\right)\Bigg]$$
$$+\Bigg[\left(t^2+qv_i^2\right)ln\left(\frac{t^2+qv_i^2}{t^2}\right)+\left(t^2+q\left(v_i^{'}\right)^2\right)ln\left(\frac{t^2+q\left(v_i^{'}\right)^2}{t^2}\right)-\left(2t^2+q\left(v_i^2+\left(v_i^{'}\right)^2\right)\right)ln\left(\frac{2t^2+q\left(v_i^2+\left(v_i^{'}\right)^2\right)}{2t^2}\right)\Bigg]$$
$$+\Bigg[\left(t^2+q\pi_i^2\right)ln\left(\frac{t^2+q\pi_i^2}{t^2}\right)+\left(t^2+q\left(\pi_i^{'}\right)^2\right)ln\left(\frac{t^2+q\left(\pi_i^{'}\right)^2}{t^2}\right)-\left(2t^2+q\left(\pi_i^2+\left(\pi_i^{'}\right)^2\right)\right)ln\left(\frac{2t^2+q\left(\pi_i^2+\left(\pi_i^{'}\right)^2\right)}{2t^2}\right)\Bigg]\Bigg\}.$$

Assume that $(\lambda_1,\lambda_2,\cdots,\lambda_q) = (\lambda_k)_{1\times q}$ is the expert weights vector. It follows from the comparison of two LPFNs that (s_t,s_0) and (s_0,s_t) are the largest and the smallest LPFN, respectively. Therefore, we can define the positive decision matrix $\left(\alpha_{ij}^{+}\right) = (s_t,s_0)_{n\times m}$ and negative decision matrix $\left(\alpha_{ij}^{-}\right) = (s_0,s_t)_{n\times m}$. According to Qi [56], a certain LPFN has more reliability when it has a larger discrimination from $\left(\alpha_{ij}^{+}\right) = (s_t,s_0)_{n\times m}$ or $\left(\alpha_{ij}^{-}\right) = (s_0,s_t)_{n\times m}$, this discrimination measure can be expressed

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left|E_C\left(\alpha_{ij}^{(k)},\alpha_{ij}^{-}\right)-E_C\left(\alpha_{ij}^{(k)},\alpha_{ij}^{+}\right)\right|$$

to depict the reliability of an IDM. As far as the expert weights are concerned, if the expert gives IDM with larger reliability, the weight should be much bigger. In order to determine the experts weights under relative weights information of experts is unknown completely, the optimal model can be constructed as follows:

$$\begin{aligned}&maxW_E(\lambda_k)\\&=\frac{1}{mn}\sum_{k=1}^{q}\lambda_k\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|E_C\left(\alpha_{ij}^{(k)},\alpha_{ij}^{-}\right)-E_C\left(\alpha_{ij}^{(k)},\alpha_{ij}^{+}\right)\right|\right)\\&s.t\begin{cases}\sum_{k=1}^{p}(\lambda_k)^2=1,\\\lambda_k\geq 0, k=1,2,\cdots,q.\end{cases}\end{aligned}$$

We can obtain expert weights λ_k^* as follows by using Lagrange multiplier method:

$$\lambda_k^* = \frac{\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left|E_C\left(\alpha_{ij}^{(k)},\alpha_{ij}^{-}\right)-E_C\left(\alpha_{ij}^{(k)},\alpha_{ij}^{+}\right)\right|}{\sqrt{\frac{1}{mn}\sum_{k=1}^{q}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|E_C\left(\alpha_{ij}^{(k)},\alpha_{ij}^{-}\right)-E_C\left(\alpha_{ij}^{(k)},\alpha_{ij}^{+}\right)\right|\right)}},\quad (k=1,2,\cdots,q)$$

We can obtain the expert weights λ_k by normalizing $\lambda_k^*\,(k=1,2,\cdots,q)$

$$\lambda_k = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\left|E_C\left(\alpha_{ij}^{(k)},\alpha_{ij}^{-}\right)-E_C\left(\alpha_{ij}^{(k)},\alpha_{ij}^{+}\right)\right|}{\sum_{k=1}^{q}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|E_C\left(\alpha_{ij}^{(k)},\alpha_{ij}^{-}\right)-E_C\left(\alpha_{ij}^{(k)},\alpha_{ij}^{+}\right)\right|\right)}. \quad (16)$$

C. DETERMINATION OF ATTRIBUTES' WEIGHTS BASED ON MAXIMIZING DEVIATIONS METHOD

MADM usually ranks and compares the comprehensive attribute values of alternatives. The smaller the difference of attribute values of all schemes under a certain attribute, the smaller the effect of the attribute on the decision-making ranking of alternatives; the greater the difference of attribute values of all alternatives under a certain attribute, the greater the effect of the attribute on the decision-making ranking of alternatives. Therefore, from the point of view of ranking alternatives, the more deviations of attribute values of alternative , the larger the weight should be given to the attributes. For the kth Decision maker e_k and attribute a_j, the deviation of alternative Ψ_i to all the other alternatives can be represented as $D_{ij}^{(k)}(w)$:

$$D_{ij}^{(k)}(w) = \sum_{l=1}^{m} d\left(\alpha_{ij}^{(k)},\alpha_{lj}^{(k)}\right)w_j,$$

where $i=1,2,\cdots,m$ and $j=1,2,\cdots,n$.

For the kth decision maker e_k and attribute a_j, the overall deviation between all alternatives and other alternatives can be expressed as

$$\begin{aligned}D_j^{(k)}(w) &= \sum_{i=1}^{m} D_{ij}^{(k)}(w)\\&=\sum_{i=1}^{m}\sum_{l=1}^{m} d\left(\alpha_{ij}^{(k)},\alpha_{lj}^{(k)}\right)w_j, (j=1,2,\cdots,n).\end{aligned}$$

Therefore, the weights of attributes can be obtained by the following optimal model:

$$\begin{aligned}maxD(w) &= \sum_{k=1}^{q}\lambda_k\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{l=1}^{m} d\left(\alpha_{ij}^{(k)},\alpha_{lj}^{(k)}\right)w_j.\\&s.t\begin{cases}\sum_{j=1}^{n}(w_j)^2=1,\\w_j\geq 0, j=1,2,\cdots,n.\end{cases}\end{aligned}$$

We can obtain attribute weights w_j^* by applying Lagrange multiplier method:

$$w_j^* = \frac{\sum_{k=1}^{q}\lambda_k\sum_{i=1}^{m}\sum_{l=1}^{m} d\left(\alpha_{ij}^{(k)},\alpha_{lj}^{(k)}\right)}{\sqrt{\sum_{j=1}^{n}\left(\sum_{k=1}^{q}\lambda_k\sum_{i=1}^{m}\sum_{l=1}^{m} d\left(\alpha_{ij}^{(k)},\alpha_{lj}^{(k)}\right)\right)^2}},\quad (j=1,2,\cdots,n).$$

We can obtain the attribute weights by normalizing $w_j^* \, (j = 1, 2, \cdots, m)$

$$w_j = \frac{\sum_{k=1}^{q} \lambda_k \sum_{i=1}^{m} \sum_{l=1}^{m} d\left(\alpha_{ij}^{(k)}, \alpha_{lj}^{(k)}\right)}{\sum_{j=1}^{n} \left(\sum_{k=1}^{q} \lambda_k \sum_{i=1}^{m} \sum_{l=1}^{m} d\left(\alpha_{ij}^{(k)}, \alpha_{lj}^{(k)}\right)\right)} \quad (17)$$

D. AN APPROACH TO LPMAGDM WITH LPI

Based on the discussed above, an approach to LPMAGDM under LPI with unknown expert weights and attribute weights will be devised in this subsection and is described as follows:

Step 1. Normalizing the original DM according to the following ways:

$$\tilde{\alpha}_{ij}^{(k)} = \left(\tilde{s}_{u_{ij}}^{(k)}, \tilde{s}_{v_{ij}}^{(k)}\right) = \begin{cases} \alpha_{ij}^{(k)} = \left(s_{u_{ij}}^{(k)}, s_{v_{ij}}^{(k)}\right), c_j \text{ is benefit type} \\ \left(\alpha_{ij}^{(k)}\right)^c = \left(s_{v_{ij}}^{(k)}, s_{u_{ij}}^{(k)}\right), c_j \text{ is cost type.} \end{cases} \quad (18)$$

The kth decision maker provides the attribute values of alternative $\Psi_i \, (i = 1, 2, \cdots, m)$ w. r. t. attribute $a_j \, (j = 1, 2, \cdots, n)$ and construct the normalized decision matrices $\tilde{R}^{(k)} = \left(\tilde{\alpha}_{ij}^{(k)}\right)_{m \times n}$.

Step 2. Determining the experts weights according to the Eq.(16).

Step 3. Determining the attributes weights according to the Eq.(17).

Step 4. Aggregating individual LPF decision matrix $\tilde{R}^{(k)} = \left(\tilde{\alpha}_{ij}^{(k)}\right)_{m \times n}$ into a collective individual LPF decision matrix $\tilde{R}^{(k)} = \left(\tilde{\alpha}_{i}^{(k)}\right)_{m \times 1}$ by Eq.(5).

Step 5. Aggregating all collective LPF decision matrix $\tilde{R}^{(k)} = \left(\tilde{\alpha}_{i}^{(k)}\right)_{m \times 1}$ into the overall LPF decision matrix $\tilde{R} = \left(\tilde{\alpha}_{i}^{(k)}\right)_{m \times 1}$ by Eq. (5).

Step 6. Calculating the ordering index by using Eq.(4).

Step 7. Ranking the alternatives according to the comparison principle, and obtaining the desirable alternative.

Step 8. End.

VII. CASE STUDY

In this section, an illustrative example which was adopt illustrative example from Garg [40] for a venture capital company plans to renewable energy projects. Assume that there are four possible projects $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ which are assessed by three experts E_1, E_2, E_3 under following four attributes:

a_1: Technical performance, that is, technical proficiency on which the project relies;

a_2: Market potential, that is, the significance of the economic and social benefits expected by the Project;

a_3: Policy environment, that is, consistency between projects and current national policies;

a_4: Investment risk, that is, uncertainty of project investment.

The experts use linguistic term collection $\mathfrak{S} = \{s_0 =$ extremely poor, $s_1 =$ very poor, $s_2 =$ poor, $s_3 =$ slightly poor, $s_4 =$ fair, $s_5 =$ slightly good, $s_6 =$ good, $s_7 =$ very good, $s_8 =$ extremely good$\}$ to evaluated potential project. Three decision matrices are given by three experts E_1, E_2, E_3 and shown in Table 1, respectively,

experts		a_1	a_2	a_3	a_4
$E^{(1)}$	Ψ_1	(s_6, s_3)	(s_3, s_1)	(s_3, s_3)	(s_3, s_6)
	Ψ_2	(s_7, s_2)	(s_6, s_2)	(s_4, s_1)	(s_2, s_4)
	Ψ_3	(s_6, s_2)	(s_5, s_3)	(s_7, s_1)	(s_3, s_4)
	Ψ_4	(s_5, s_3)	(s_7, s_2)	(s_4, s_3)	(s_1, s_6)
$E^{(2)}$	Ψ_1	(s_6, s_1)	(s_4, s_5)	(s_5, s_2)	(s_3, s_5)
	Ψ_2	(s_7, s_2)	(s_5, s_1)	(s_7, s_2)	(s_4, s_3)
	Ψ_3	(s_5, s_3)	(s_6, s_3)	(s_6, s_3)	(s_4, s_4)
	Ψ_4	(s_5, s_2)	(s_4, s_3)	(s_5, s_1)	(s_5, s_3)
$E^{(3)}$	Ψ_1	(s_5, s_3)	(s_4, s_4)	(s_7, s_2)	(s_4, s_2)
	Ψ_2	(s_7, s_1)	(s_7, s_1)	(s_6, s_3)	(s_1, s_4)
	Ψ_3	(s_5, s_2)	(s_4, s_3)	(s_5, s_2)	(s_3, s_6)
	Ψ_4	(s_6, s_3)	(s_5, s_3)	(s_5, s_4)	(s_2, s_5)

TABLE 1. Individual linguistic Pythagorean DMs given by $E^{(k)} (k = 1, 2, 3)$

A. DECISION-MAKING PROCESS FOR SELECTING THE BEST PROJECT

Step 1. Normalizing the original DM. Because the attribute a_4 is cost-type, so we need to normalize the attribute values of a_4 by using Eq.(43), and the normalized decision matrices listed as follows (Table 2):

experts		a_1	a_2	a_3	a_4
$E^{(1)}$	Ψ_1	(s_6, s_3)	(s_3, s_1)	(s_3, s_3)	(s_6, s_3)
	Ψ_2	(s_7, s_2)	(s_6, s_2)	(s_4, s_1)	(s_4, s_2)
	Ψ_3	(s_6, s_2)	(s_5, s_3)	(s_7, s_1)	(s_4, s_3)
	Ψ_4	(s_5, s_3)	(s_7, s_2)	(s_4, s_3)	(s_6, s_1)
$E^{(2)}$	Ψ_1	(s_6, s_1)	(s_4, s_5)	(s_5, s_2)	(s_5, s_3)
	Ψ_2	(s_7, s_2)	(s_5, s_1)	(s_7, s_2)	(s_3, s_4)
	Ψ_3	(s_5, s_3)	(s_6, s_3)	(s_6, s_3)	(s_4, s_4)
	Ψ_4	(s_5, s_2)	(s_4, s_3)	(s_5, s_1)	(s_3, s_5)
$E^{(3)}$	Ψ_1	(s_5, s_3)	(s_4, s_4)	(s_7, s_2)	(s_2, s_4)
	Ψ_2	(s_7, s_1)	(s_7, s_1)	(s_6, s_3)	(s_4, s_1)
	Ψ_3	(s_5, s_2)	(s_4, s_3)	(s_5, s_2)	(s_6, s_3)
	Ψ_4	(s_6, s_3)	(s_5, s_3)	(s_5, s_4)	(s_5, s_2)

TABLE 2. Normalized Individual linguistic Pythagorean DMs given by $E^{(k)} (k = 1, 2, 3)$

Step 2. Determining the experts weights. According to the Eq.(16), taking $q = 0.5$ in cross-entropy, we obtain the expert weight vector is (0.3465, 0.3179, 0.3356).

Step 3. Determining the attributes weights. According to the Eq.(17), taking $p = 0.5$ in Eq.(3), we obtain the attribute weight vector is (0.2086, 0.2493, 0.2670, 0.2301).

experts	Ψ_1	Ψ_2	Ψ_3	Ψ_4
$E^{(1)}$	$(s_{4.806}, s_{2.171})$	$(s_{5.645}, s_{1.662})$	$(s_{5.867}, s_{2.056})$	$(s_{5.93}, s_{2.068})$
$E^{(2)}$	$(s_{5.031}, s_{2.488})$	$(s_{6.1}, s_{2.244})$	$(s_{5.48}, s_{3.205})$	$(s_{4.582}, s_{2.312})$
$E^{(3)}$	$(s_{5.341}, s_{3.131})$	$(s_{6.391}, s_{1.341})$	$(s_{5.058}, s_{2.474})$	$(s_{5.254}, s_{2.951})$

TABLE 3. Collective individual LPF DMs of $E^{(k)}(k = 1, 2, 3)$

Step 4. Aggregating individual LPF DM $\tilde{R}^{(k)} = (\tilde{\alpha}_{ij}^{(k)})_{m\times n}$ into a collective individual LPF DM $\tilde{R}^{(k)} = (\tilde{\alpha}_{i}^{(k)})_{m\times 1}$ by Eq.(5).

Step 5. Aggregating all collective LPF DM $\tilde{R}^{(k)} = (\tilde{\alpha}_{i}^{(k)})_{m\times 1}$ into the overall LPF DM $\tilde{R} = (\tilde{\alpha}_{i}^{(k)})_{m\times 1}$ by Eq.(5).

Step 6. Calculating the ordering index by using Eq.(4),

$R(\Psi_1) = 0.4147, R(\Psi_2) = 0.2785, R(\Psi_3) = 0.3465, R(\Psi_4) = 0.3756.$

Step 7. Ranking the alternatives according to the comparison principle, the rank of alternatives is $\Psi_2\Psi_3\Psi_4\Psi_1$, and so Ψ_2 is the best alternative.

B. DISCUSSIONS ON PARAMETERS

In this section, we will discuss the effect of parameter changes on the results.

(1) Effect of parameter changes in cross-entropy on the results under $p = 0.5$ in generalized indeterminacy degree-preference distance measure and $\lambda = 1$.

Although the values of cross-entropy will increases as $q(q > 0)$ increases, the experts' weights determined by cross-entropy almost unchanged, which lead to the same rank under different parameter q. This point will be showed from above Table that the ranking of alternatives will remain unchanged.

(2) Effect of parameter changes in generalized indeterminacy degree-preference distance on the results under $q = 1$ in cross-entropy.

Although the values of cross-entropy will increases as $q(q > 0)$ increases, the experts' weights determined by cross-entropy almost unchanged, which lead to the same rank under different parameter q. This point will be showed from above Table that the ranking of alternatives will remain unchanged.

(3)Effect of parameter Ψ changes on the results under $q = 0.5$ in cross-entropy and $p = 0.5$ in generalized indeterminacy degree-preference distance.

C. COMPARISONS AND ANALYSES

In what following, the analyses of proposed method will be carried out and comparisons with other existed method also be conducted.

Firstly, we use some special cased from the proposed five operators to rank alternatives when $\lambda = 1, p = q = 0.5$, which are shown as Table 8.

It is seen from Table 8 that the ranking order of alternatives are the same.

Secondly, we use our proposed methods to solve the MAGDM problem [40] under the experts weight completely known and the attribute weights is (0.3, 0.2, 0.1, 0.4). The ranking order of alternatives are shown as Table 9.

• (1) Garg [40] introduced the LPFNs and also given some operational laws of LPFNs. In present work, we firstly extended TN and TC, and then give five special cases of TN and TC. On the basis of them, the general forms of operational laws are also given. The operational law introduced by Garg [40] is only the special case when the extended TN and TC are T_A and S_A.

• (2) As far as the aggregation operators are concerned. Garg [40] introduced the LPFWA operator and LPFWG operator to deal with LPFI. In present work, GLPFWA was proposed based on extended TN and TC, some different types aggregation operators are obtained with different parameters, extended TN and extended TC. However, some existing operators such as LPFWA and LPFWG are the special cases of GLPFWA. The details have been investigated in detail in Section 7.

• (3) As far as decision methods are concerned. Garg's MAGDM method [40] is based on the LPFWA and LPFWGA by considering the experts' weights are unknown, but the attribute weights are also known. Although the proposed MAGDM method is also based on GLPFWA, there are two significant differences between them: on the one hand, the proposed MAGDM method is more general than Garg [40] because the proposed approach uses the some parameters and extended TN and extended TC; on the other hand, the proposed approach addressed the decision problems in which the experts weights and attribute weights are all unknown.

Thirdly, although some linguistic decision making approaches have been developed, such as LIFDM approach [34], MAGDM approach based on I2LI [47], they only addressing the some decision making problems with LIFI, the proposed MAGDM approach not only solve intuitionistic fuzzy decision making problem, but also solve some decision making problems with LPFI which is not addressed by some existed MAGDM approaches.

Finally, as far as the LPTNs and LPTCs are concerned, although Liu [27] extended archimedean TN and TC from $[0, 1]$ to $[0, t]$, which can deal with some decision making problems with LIFI and it really doesn't work for some problems with LPFI. In this work, we proposed more general extended TN and extended TC which can deal with LPFI by some new additive generators.

Ψ_1	Ψ_2	Ψ_3	Ψ_4
$(s_{5.0687}, s_{2.5635})$	$(s_{6.0685}, s_{1.9129})$	$(s_{5.5007}, s_{2.5193})$	$(s_{5.3477}, s_{2.4142})$

TABLE 4. Overall LPF Decision matrix

Parameter	Ordering index of $\Psi_i (i = 1, 2, 3, 4)$	Ranking order
$q = 1$	$R(\Psi_1) = 0.4147, R(\Psi_2) = 0.2806, R(\Psi_3) = 0.3465, R(\Psi_4) = 0.3756$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$q = 2$	$R(\Psi_1) = 0.4146, R(\Psi_2) = 0.2785, R(\Psi_3) = 0.3465, R(\Psi_4) = 0.3757$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$q = 5$	$R(\Psi_1) = 0.4146, R(\Psi_2) = 0.2784, R(\Psi_3) = 0.3465, R(\Psi_4) = 0.3759$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$q = 10$	$R(\Psi_1) = 0.4147, R(\Psi_2) = 0.2784, R(\Psi_3) = 0.3465, R(\Psi_4) = 0.3759$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$q = 100$	$R(\Psi_1) = 0.4147, R(\Psi_2) = 0.2785, R(\Psi_3) = 0.3464, R(\Psi_4) = 0.3759$	$\Psi_2\Psi_3\Psi_4\Psi_1$

TABLE 5. Overall LPF DM

Parameter	Ordering index of $\Psi_i (i = 1, 2, 3, 4)$	Ranking order
$p = 0$	$R(\Psi_1) = 0.4551, R(\Psi_2) = 0.2957, R(\Psi_3) = 0.3843, R(\Psi_4) = 0.4094$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$p = 0.1$	$R(\Psi_1) = 0.4473, R(\Psi_2) = 0.2928, R(\Psi_3) = 0.3771, R(\Psi_4) = 0.4029$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$p = 0.3$	$R(\Psi_1) = 0.4313, R(\Psi_2) = 0.2869, R(\Psi_3) = 0.3612, R(\Psi_4) = 0.3895$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$p = 0.5$	$R(\Psi_1) = 0.4147, R(\Psi_2) = 0.2806, R(\Psi_3) = 0.3465, R(\Psi_4) = 0.3756$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$p = 0.7$	$R(\Psi_1) = 0.3973, R(\Psi_2) = 0.2740, R(\Psi_3) = 0.3302, R(\Psi_4) = 0.3613$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$p = 0.9$	$R(\Psi_1) = 0.3790, R(\Psi_2) = 0.2665, R(\Psi_3) = 0.3131, R(\Psi_4) = 0.3464$	$\Psi_2\Psi_3\Psi_4\Psi_1$

TABLE 6. Overall LPF DM

Parameter	Ordering index of $\Psi_i (i = 1, 2, 3, 4)$	Ranking order
$\lambda = 0.05$	$R(\Psi_1) = 0.4533, R(\Psi_2) = 0.3068, R(\Psi_3) = 0.3613, R(\Psi_4) = 0.4018$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$\lambda = 0.5$	$R(\Psi_1) = 0.352, R(\Psi_2) = 0.2942, R(\Psi_3) = 0.3546, R(\Psi_4) = 0.3933$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$\lambda = 1$	$R(\Psi_1) = 0.4147, R(\Psi_2) = 0.2807, R(\Psi_3) = 0.3465, R(\Psi_4) = 0.3829$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$\lambda = 5$	$R(\Psi_1) = 0.2982, R(\Psi_2) = 0.2110, R(\Psi_3) = 0.2806, R(\Psi_4) = 0.2974$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$\lambda = 10$	$R(\Psi_1) = 0.2334, R(\Psi_2) = 0.1754, R(\Psi_3) = 0.2268, R(\Psi_4) = 0.2311$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$\lambda = 50$	$R(\Psi_1) = 0.1540, R(\Psi_2) = 0.1440, R(\Psi_3) = 0.1506, R(\Psi_4) = 0.1527$	$\Psi_2\Psi_3\Psi_4\Psi_1$
$\lambda = 100$	$R(\Psi_1) = 0.1516, R(\Psi_2) = 0.1406, R(\Psi_3) = 0.1432, R(\Psi_4) = 0.1446$	$\Psi_2\Psi_3\Psi_4\Psi_1$

TABLE 7. Overall LPF DM

Operators	Ordering index of $\Psi_i (i = 1, 2, 3, 4)$	Ranking order
LPFWA	$R(\Psi_1) = 0.4147, R(\Psi_2) = 0.2785, R(\Psi_3) = 0.3465, R(\Psi_4) = 0.3756$	$\Psi_2\Psi_3\Psi_4\Psi_1$
LPFEWA	$R(\Psi_1) = 0.4092, R(\Psi_2) = 0.2964, R(\Psi_3) = 0.3426, R(\Psi_4) = 0.3833$	$\Psi_2\Psi_3\Psi_4\Psi_1$
LPFHWA	$R(\Psi_1) = 0.1956, R(\Psi_2) = 0.1147, R(\Psi_3) = 0.1515, R(\Psi_4) = 0.1794$	$\Psi_2\Psi_3\Psi_4\Psi_1$
LPFFWA	$R(\Psi_1) = 0.4217, R(\Psi_2) = 0.2874, R(\Psi_3) = 0.3501, R(\Psi_4) = 0.3860$	$\Psi_2\Psi_3\Psi_4\Psi_1$
LPFDWA	$R(\Psi_1) = 0.3315, R(\Psi_2) = 0.2029, R(\Psi_3) = 0.2994, R(\Psi_4) = 0.4904$	$\Psi_2\Psi_3\Psi_4\Psi_1$

TABLE 8. Ranking orders from different operators

Methods	Ordering index of $\Psi_i(i=1,2,3,4)$	Ranking order
GLPFWA	$R(\Psi_1)=0.6043, R(\Psi_2)=0.6909, R(\Psi_3)=0.6036, R(\Psi_4)=0.3607$	$\Psi_4\Psi_3\Psi_1\Psi_2$
GLPFEWA	$R(\Psi_1)=0.6183, R(\Psi_2)=0.6918, R(\Psi_3)=0.6276, R(\Psi_4)=0.3669$	$\Psi_4\Psi_3\Psi_1\Psi_2$
GLPFHWA	$R(\Psi_1)=0.4151, R(\Psi_2)=0.5225, R(\Psi_3)=0.3685, R(\Psi_4)=0.1644$	$\Psi_4\Psi_3\Psi_1\Psi_2$
GLPFFWA	$R(\Psi_1)=0.6343, R(\Psi_2)=0.7031, R(\Psi_3)=0.5957, R(\Psi_4)=0.3546$	$\Psi_4\Psi_3\Psi_1\Psi_2$
GLPFDWA	$R(\Psi_1)=0.5182, R(\Psi_2)=0.7186, R(\Psi_3)=0.4838, R(\Psi_4)=0.3187$	$\Psi_4\Psi_3\Psi_1\Psi_2$
Garg's Method [40]		$\Psi_4\Psi_3\Psi_1\Psi_2$

TABLE 9. Ranking orders of our proposed method

VIII. CONCLUSIONS

This work aims to build the new approach for MAGDM with unknown experts weight and attribute weights. Firstly, the approach is given to rank the alternatives based on generalized indeterminacy degree-preference distance under LPFI, the new versions of TNs and TCs named LPTNs and LPTCs are given to handle the LPFI, and some special cases for LPTNs and LPTCs are obtained to deal with PFI; Secondly, we introduce the generalized linguistic Pythagorean fuzzy average aggregation operator (GPFAA) based on LPTNs and LPTCs along with their properties are also investigated; Furthermore, we analyse the GPFAA operators and some special cases are obtained when LPTNs and LPTCs take some special TNs and TCs; Finally, a MAGDM approach based on some LPTNs and LPTCs is established, before building the MAGDM approach, we define new cross-entropy to determine the experts's weights and use the maximizing deviation to determine the attributes' weights, consequently, a practical example is provided to show the validity and advantages of the proposed method.

The use of several LTSs in fuzzy linguistic modeling is allowed in multigranular fuzzy linguistic modeling, which has been frequently used in GDM field due to its capability of allowing each expert to express his/her preferences using his/her own LTS. A new linguistic computational model is defined by Zhang [57] to manage multigranular linguistic distribution assessments for its application to large-scale MAGDM problems with linguistic information. In our future research, by means of academic thought of TODIM method based on unbalanced HFLTs [58], we will study the TODIM method for large-scale MAGDM problems with unbalanced linguistic information or multigranular linguistic information.

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