



## Determining the spring constant of arbitrarily shaped cantilevers in viscous environments

Farokh Payam, A., Trewby, W., & Voitchovsky, K. (2018). Determining the spring constant of arbitrarily shaped cantilevers in viscous environments. *Applied Physics Letters*, 112(8), 1-4. Article 083101. Advance online publication. <https://doi.org/10.1063/1.5009071>

[Link to publication record in Ulster University Research Portal](#)

**Published in:**  
Applied Physics Letters

**Publication Status:**  
Published online: 20/02/2018

**DOI:**  
[10.1063/1.5009071](https://doi.org/10.1063/1.5009071)

**Document Version**  
Author Accepted version

### General rights

Copyright for the publications made accessible via Ulster University's Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

### Take down policy

The Research Portal is Ulster University's institutional repository that provides access to Ulster's research outputs. Every effort has been made to ensure that content in the Research Portal does not infringe any person's rights, or applicable UK laws. If you discover content in the Research Portal that you believe breaches copyright or violates any law, please contact [pure-support@ulster.ac.uk](mailto:pure-support@ulster.ac.uk).

# Determining the spring constant of arbitrary shaped cantilevers in viscous environments

A. F. Payam<sup>1,a)</sup>, W. Trewby<sup>1, a)</sup> and K. Voïtchovsky<sup>1,b)</sup>

<sup>1</sup>*Department of Physics, Durham University, Durham, DH1 3LE, UK*

Accurate calibration of the flexural spring constant of microcantilevers is crucial for sensing devices, microactuators and for atomic force microscopy (AFM). Existing **theoretical and quantitative** methods rely on precise knowledge of cantilever geometry, or make significant simplifications. Here we develop a simple equation to calculate the flexural spring constants of arbitrarily shaped cantilevers in fluid. Our approach, verified here with AFM, only requires the measurement of two resonance frequencies of the cantilever in air and in a liquid, with no need for additional input or knowledge about the system. We validate the method with cantilevers of different shapes and compare its predictions with existing models. We also show how the method's accuracy can be considerably improved, especially in more viscous liquids, if the effective width of the cantilever is known. Significantly, the developed equations can be extended to calculate the spring constants of the cantilever's higher Eigenmodes.

Microcantilevers constitute the backbone of a wide range of technologies, from actuators in MEMS [1], to sensors [2], lab-on chip technology [3], and for atomic force microscopy (AFM) [4]. Microcantilever-based measurements of forces with pico-Newton resolution and of displacements down to the Ångstrom level are now commonplace. Most applications rely on small relative displacements or bending of the lever, which can generally be modelled by a linear spring with a single flexural spring constant,  $k_f$ . Precise knowledge of  $k_f$  is hence key to achieving accurate and reproducible results, and considerable efforts have been dedicated to the modelling of cantilevers' motion [5] and to the development of calibration procedures [6]. The task is, however, highly challenging because accurate predictions require precise knowledge of the cantilever's geometrical and physical characteristics, something far from trivial given manufacturing variability at that scale. Additionally, no single model works for all cantilever geometries, let alone unusual and arbitrary shapes, unless considerable simplifications are made [7]. To add to this complexity, the dynamical properties of cantilevers on which most models rely strongly depend on the viscoelastic properties of the immediate environment.

In the field of AFM, the need for accurate  $k_f$  characterization is a central issue given the use of microcantilevers to measure minute molecular [8–10] or interfacial forces [11–13], often close to the thermal limit. It also feeds into the problem of force-reconstruction, where the spatial landscape of a force potential is

---

<sup>a)</sup> A. F. Payam and W. Trewby contributed equally to this work

<sup>b)</sup> Author to whom correspondence should be addressed. Electronic mail: [kislon.voitchovsky@durham.ac.uk](mailto:kislon.voitchovsky@durham.ac.uk)

calculated from dynamical measurements with a vibrating cantilever [14–19]. Perhaps unsurprisingly, many AFM-based methods have been proposed to estimate  $k_f$  based on the dynamic motion of the cantilever, typically measured by a laser focused near the cantilever’s free extremity [6,20–30]. To date, the most common methods are the so-called thermal noise method [21,31,32] and the Sader method [18,22,23,30,33] – a comprehensive review of all the most methods available can be found in ref. [6].

The thermal noise method is derived from equipartition theory and requires knowledge of the frequency-dependent response of the cantilever to thermal fluctuations in the surrounding environment (i.e. the thermal spectrum, see fig. 1 (a)) [21,32]. If the inverse optical lever sensitivity (invOLS) of the cantilever-laser system is known, fitting of the thermal spectrum can be used to find  $k_f$  for any resonant mode of the cantilever. The method’s accuracy depends on mechanical noises as well as the white noise from the environment. The result is also highly sensitive to the choice of interval used to fit the relevant portion of the thermal spectrum. Significantly, measurement of the invOLS usually requires bending the cantilever on a hard substrate, a procedure that can damage or permanently alter the measuring tip. This can, in principle, be avoided by calibrating the invOLS after an experiment, at the cost of uncertainties in the forces applied to the sample of interest.

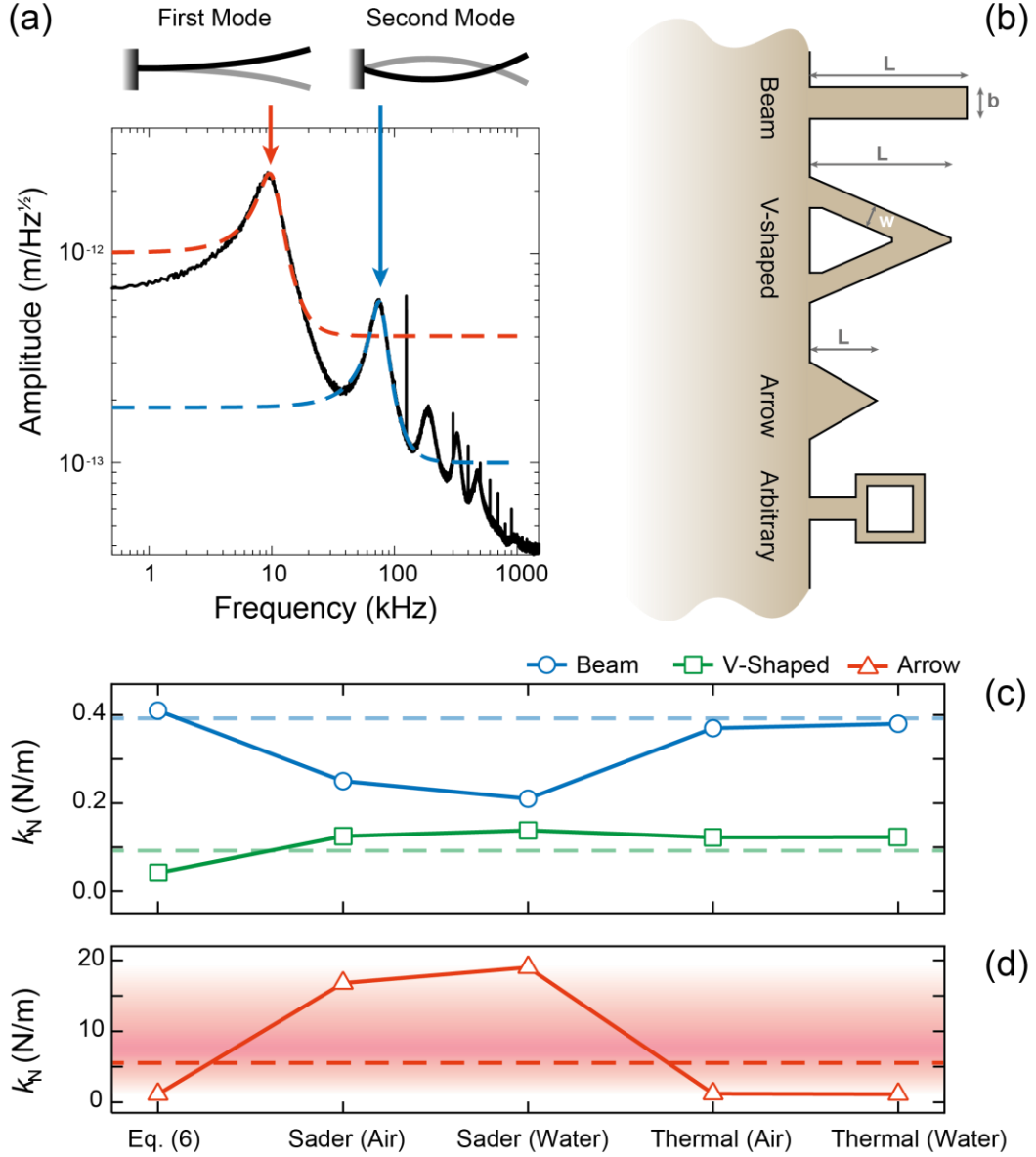


FIG. 1: Calculation of the flexural spring constants from cantilever dynamics. (a) Example thermal spectrum recorded with a V-shaped cantilever immersed in water. The first two resonance peaks are highlighted, as well as the corresponding modes of oscillation. Simple harmonic oscillator model predictions are shown for the first and second modes in red and blue dashes respectively. (b) Examples of differently shaped cantilevers and the relevant dimensions used in theoretical models. The upper three types will be investigated in this study. The lower cantilever exemplifies a shape that would represent a challenge for traditional methods. (c) Flexural spring constant,  $k_f$ , predicted by our eq. (6), the Sader method and the thermal method for a beam and V-shaped cantilever in air and water. Nominal manufacturer's values are highlighted with dashed lines. In general, eq. (6) produces results consistent with the manufacturer's values and those measured via the thermal method. In the case of the beam, eq. (6) appears to give more robust results than the Sader model in both air and water. (d) Flexural spring constants predicted by the various models for an arrow-shaped cantilever. Here, there is no good agreement between the (poorly defined) nominal stiffness and any model, but eq. (6) again agrees well with the thermal method. The red gradient here reflects the manufacturer's broad nominal stiffness range ( $k_f \sim 6$  N/m typical, with  $1.5$  N/m  $< k_f < 20.0$  N/m). Calculated errors for eq. 6 are smaller than the data markers.

The Sader method, developed by J. Sader [18,22,23,30,33], is also based on the thermal spectrum of the cantilever but relies on a more sophisticated fitting procedure that takes into account the fundamental resonance frequency of the cantilever, its quality (Q) factor, geometrical shape and hydrodynamic function. The method works for

cantilevers with rectangular or V-shaped geometries [22] (see fig. 1 (b)). It can be adapted to other shapes but requires the input of cantilever-dependent parameters that are not readily measurable [22].

Both the thermal noise and Sader methods have become standard benchmarks in the field; they can be implemented in air or liquid environments, but often lead to different results due to strongly enhanced – and often tip-sample separation dependent – damping in fluid environments. Prediction errors can become severe in highly viscous environments, partly due to difficulties in accurately measuring a quality factor. Moreover, for cantilevers with non-standard geometry (fig. 1 (b)) the Sader method requires further adjustments [22].

Here, we propose a novel method for calibrating the flexural stiffness of cantilevers with arbitrary shapes in viscous fluids. Our approach requires only knowledge of the cantilever’s length and its first two resonant frequencies in air and liquid. There is no dependence on the cantilever’s quality factor and no need for invOLS calibration, and results in air or viscous liquids are comparable or more accurate than with the existing methods. If the effective width of the cantilever is known, we show that the accuracy of the predictions can be further improved by up to an order of magnitude over established methods with the knowledge of only first resonance frequencies of cantilever in air or desired liquid.

Our method begins from the dynamic motion of the cantilever, based on the Euler-Bernoulli partial differential equation [34]:

$$EI \frac{\partial^4}{\partial x^4} W(x, t) + \rho_c b h \frac{\partial^2}{\partial t^2} W(x, t) = F_{\text{exc}} + F_h, \quad (1)$$

where  $E$  is the cantilever’s Young-modulus,  $I$  is its rotary inertia,  $\rho_c$  is the cantilever density and  $b$  and  $h$  are the width and thickness of the cantilever, respectively.  $W(x, t)$  is the time-dependent displacement of the cantilever,  $F_{\text{exc}}$  is the excitation force and  $F_h$  is the hydrodynamic force which can be described by a separate added mass and damping stiffness. Considering the added mass and damping stiffness per unit length of the cantilever [34–36] and assuming a hydrodynamic function characterized by two real ( $a_1$  and  $a_2$ ) and two imaginary ( $b_1$  and  $b_2$ ) regression coefficients [35,37,38] we can relate the resonance frequencies of the microcantilever in air,  $\omega_{an}$ , and in an arbitrary fluid,  $\omega_{fn}$  [39], for any given mode  $n$ :

$$\omega_{fn}^2 \left[ \frac{\pi a_1 \rho_f b}{4 \rho_c h} + 1 \right] + \omega_{fn}^{3/2} \left[ \frac{\pi a_2 \sqrt{\eta \rho_f}}{2 \rho_c h} \right] = \omega_{an}^2, \quad (2)$$

where  $\rho_f$  and  $\eta$  are the density and the viscosity of the fluid respectively.

The hydrodynamic coefficients  $a_i$  and  $b_i$  are independent of the cantilever characteristics or the medium in which it operates, and have been evaluated elsewhere and do not change between cantilevers [35,37,38]. After measuring two resonance frequencies of the cantilever in both air and a liquid environment from the thermal spectrum (fig. 1 (a)), the areal mass density,  $\rho_c h$ , and width,  $b$ , can be obtained from equation (2):

$$\widehat{\rho_c h} = \frac{\pi a_2 \sqrt{\rho_f \eta}}{2} \frac{\omega_{f1}^{3/2} \omega_{f2}^{3/2} (\sqrt{\omega_{f2}} - \sqrt{\omega_{f1}})}{(\omega_{a1}^2 - \omega_{f1}^2) \omega_{f2}^2 - (\omega_{a2}^2 - \omega_{f2}^2) \omega_{f1}^2}, \quad (3)$$

$$\widehat{b} = \frac{2a_2 \sqrt{\eta}}{a_1 \sqrt{\rho_f \omega_{f1}}} \left( \frac{\omega_{f2}^{3/2} (\sqrt{\omega_{f2}} - \sqrt{\omega_{f1}}) (\omega_{a1}^2 - \omega_{f1}^2)}{(\omega_{a1}^2 - \omega_{f1}^2) \omega_{f2}^2 - (\omega_{a2}^2 - \omega_{f2}^2) \omega_{f1}^2} - 1 \right). \quad (4)$$

Here, the carat denotes a calculated value and the indices  $a$  and  $f$  refer to the Eigenfrequencies of the cantilever in air and in a fluid respectively. The flexural spring constant,  $k_{fn}$ , of each mode of the cantilever is related to the effective mass of that mode,  $m_n$ , through the relation  $k_{fn} = m_n \omega_{an}^2$  [40], and the effective mass can be found from the actual mass,  $m_c$ , or the cantilever's geometrical parameters by [40]:

$$m_n = \frac{1}{4} m_c = \frac{1}{4} \rho_c h b L, \quad (5)$$

where  $L$  represents the cantilever length. By combining eq.s (3), (4) and (5), we obtain the following expression for the cantilever's spring constant:

$$k_{f1} = \frac{1}{4} \widehat{\rho_c h} \widehat{b} L \omega_{a1}^2. \quad (6)$$

Fig. 1 (c) compares  $k_f$  predictions obtained from eq. (6), the thermal and Sader methods for the beam (RC800 PSA, Olympus) and V-shaped (TR400 PB, Olympus) cantilevers, in air and in water. For the beam cantilever we use Sader's equation (1) in ref. [22] and for the V-shaped cantilever we use the adapted equation (8) in the same paper. In each case, the cantilever's nominal values are shown as dashed lines for reference. The predictions obtained from eq. (6) broadly agree with the thermal method, and are at least as accurate as the Sader method in most cases if the nominal value is taken as a reference point. We note that there is no independent measurement of cantilever stiffness here – even manufacturer's values are typically given with large uncertainties – and so there is no formal measure of accuracy for our model. The proximity of the thermal method and nominal values does imply that our results are accurate. However, unlike the thermal method, eq. (6) does *not* require invOLS calibration with potentially damaging tip-sample contact. Further validation of our model for cantilevers of different geometries and stiffness are shown in supplementary table S1.

For arrow-shaped cantilevers (Arrow UHF AuD, Nanoworld), the parallel beam approximation developed by Sader cannot be easily adapted and so we make use of equation (1) in ref. [22]. We again find an excellent agreement between our eq. (6) and the thermal method (less than 7%, see fig. 1 (d)), demonstrating the validity of the approach.

There is however one caveat to eq. (6): the viscosity of the environment in which the cantilever operates. Most models that use dynamical measurements to find a value for  $k_f$  tend to fail when the measurements are conducted in highly viscous environments and eq. (6) is no exception. The results presented in fig. 1 are based on measurements conducted in relatively low viscosity environment – air and pure water. However, when working in more viscous liquids and especially with softer cantilevers, the quality of the predictions progressively decreases (fig. 2). This is a problem for applications such as viscometry or biosensing [39] where microcantilevers are used in non-Newtonian bodily fluids. To overcome this issue, we developed a more accurate equation for predicting  $k_f$ . This comes at the cost of an extra parameter needed as an input: the width  $b$  of the cantilever, or an effective width for non-rectangular cantilevers. If  $b$  is known, the areal mass density of cantilever  $\widehat{\rho_c h}$  becomes:

$$\widehat{\rho_c h} = \frac{\omega_{f1}^2 \pi a_1 \rho_f b + 2\omega_{f1}^{3/2} \pi a_2 \sqrt{\rho_f \eta}}{4(\omega_{a1}^2 - \omega_{f1}^2)}, \quad (7)$$

which, using eq. (5), yields the following expression for  $k_f$ :

$$k_{f1} = \frac{\omega_{f1}^2 \pi a_1 \rho_f b + 2\omega_{f1}^{3/2} \pi a_2 \sqrt{\rho_f \eta}}{16(\omega_{a1}^2 - \omega_{f1}^2)} b L \omega_{a1}^2, \quad (8)$$

Significantly, eq. (8) does not depend on the Q factor of the cantilever, making it less sensitive to the difficulties in measuring Q accurately in viscous environments. We also note that eq. (8) only requires the frequency of the first resonance of the cantilever, another advantage in viscous media. We tested eq. (8) in five additional fluids of varying viscosity; isopropanol, acetone, butanol, decane and hexanol (Sigma-Aldrich, Dorset, UK, purity > 99%). The measurements were conducted with a rectangular cantilever (Olympus RC800 PSA) and compared with predictions from eq. (6), Sader and the thermal method; the results can be seen in fig. 2.

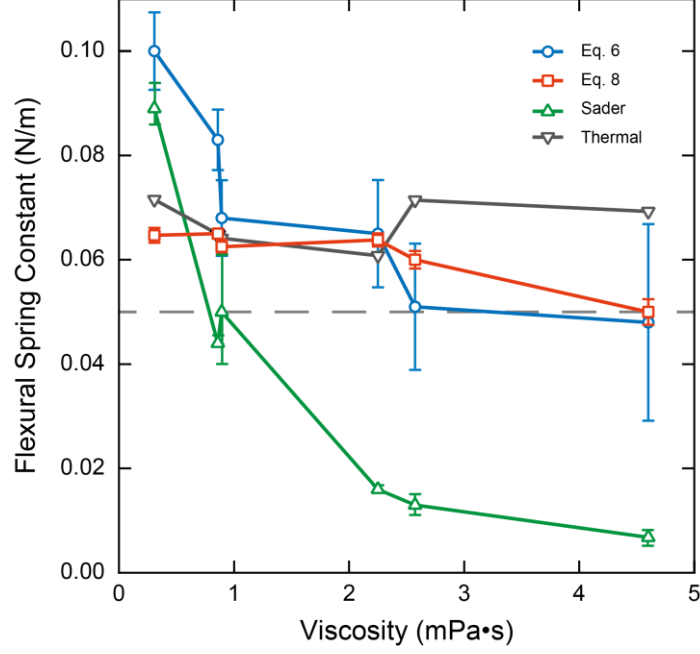


FIG. 2: Assessment of the impact of the surrounding fluid’s viscosity on the predicted spring constant of a beam cantilever. The thermal method is reasonably constant with viscosity as expected, but deviates from the nominal manufacturer’s value (dashed line) of 0.05 N/m. In contrast, both our eq. (6) and the Sader method vary significantly, decreasing as the viscosity increases. Eq. (8) performs much better, returning a stable value for  $k_f$  that agrees with the thermal method and has reduced errors at all but the highest viscosities.

The results show that the stiffness as calculated via eq. (8) is much less sensitive to viscosity than the other methods, whereas eq. (6) and the thermal method vary respectively by around 20% and the Sader method fails significantly in highly viscous mediums, likely related to the significant decrease of quality factor and errors in the measurement of quality factor in high viscous environments. This demonstrates that eq. (8) provides the most reliable model for calculating  $k_f$ , particularly when operating in viscous environments

In this paper, we propose a novel approach to determine the spring constant of a cantilever based solely on the measurement of its two first Eigenfrequencies. The method does not require any knowledge about cantilever characteristics, making it particularly useful calibration of systems where accurate determination of the calibration geometry or shape is not possible. Significantly, comparison with existing popular methods show our approach to provide similar or better results. We show that if the width of the cantilever is known, the quality of the prediction can be further improved, especially in viscous fluids where other methods tend to fail. Our equations can also be extended to determine spring constants of higher Eigenmodes of vibrating cantilevers, a key to multimodal measurements including in the fast growing field of multifrequency AFM [41,42].



See Supplementary Material for a list of the measured parameters used in Fig. 1, additional measurements with different cantilevers in viscous liquids and a detailed description of the experimental methodology, including error calculations.

This work was supported by the Engineering and Physical Sciences Research Council (Grants 1452230), the Biotechnology and Biological Science Research Council (grant BB/M024830/1) and the European Council (FP7 CIG 631186). Insightful discussions with Clodomiro Cafolla are gratefully acknowledged.

## References

- [1] S. Beeby, *MEMS Mechanical Sensors* (Artech House, 2004).
- [2] R. Datar, S. Kim, S. Jeon, P. Hesketh, S. Manalis, A. Boisen, and T. Thundat, *MRS Bull.* **34**, 449 (2009).
- [3] N. McLoughlin, S. L. Lee, and G. Hahner, *Appl. Phys. Lett.* **89**, 1 (2006).
- [4] F. J. Giessibl, *Rev. Mod. Phys.* **75**, 949 (2003).
- [5] A. Raman, J. Melcher, and R. Tung, *Nanotoday* **3**, 20 (2008).
- [6] C. T. Gibson, D. A. Smith, and C. J. Roberts, *Nanotechnology* **16**, 234 (2005).
- [7] J. P. Cleveland, S. Manne, D. Bocek, and P. K. Hansma, *Rev. Sci. Instrum.* **64**, 403 (1993).
- [8] K. Voitchovsky, J. J. Kuna, S. A. Contera, E. Tosatti, and F. Stellacci, *Nat. Nanotechnol.* **5**, 401 (2010).
- [9] D. Ortiz-Young, H.-C. Chiu, S. Kim, K. Voitchovsky, and E. Riedo, *Nat. Commun.* **4**, 2482 (2013).
- [10] K. Voitchovsky, *Nanoscale* **8**, 17472 (2016).
- [11] R. Garcia and R. Perez, *Surf. Sci. Rep.* **47**, 197 (2002).
- [12] A. Elbourne, K. Voitchovsky, G. G. Warr, and R. Atkin, *Chem. Sci.* **6**, 527 (2015).
- [13] J. J. Kuna, K. Voitchovsky, C. Singh, H. Jiang, S. Mwenifumbo, P. K. Ghorai, M. M. Stevens, S. C. Glotzer, and F. Stellacci, *Nat. Mater.* **8**, 837 (2009).

- [14] H. Hölscher, Appl. Phys. Lett. **89**, 123109 (2006).
- [15] A. F. Payam, D. Martin-Jimenez, and R. Garcia, Nanotechnology **26**, 185706 (2015).
- [16] S. Hu and A. Raman, Nanotechnology **19**, 375704 (2008).
- [17] A. J. Katan, M. H. van Es, and T. H. Oosterkamp, Nanotechnology **20**, 165703 (2009).
- [18] J. E. Sader and S. P. Jarvis, Appl. Phys. Lett. **84**, 1801 (2004).
- [19] J. E. Sader, T. Uchihashi, M. J. Higgins, A. Farrell, Y. Nakayama, and S. P. Jarvis, Nanotechnology **16**, S94 (2005).
- [20] C. T. Gibson, G. S. Watson, and S. Myhra, Nanotechnology **7**, 259 (1996).
- [21] R L'evy and M Maaloum, Nanotechnology **13**, 33 (2002).
- [22] J. E. Sader, J. A. Sanelli, B. D. Adamson, J. P. Monty, X. Wei, S. A. Crawford, J. R. Friend, I. Marusic, P. Mulvaney, and E. J. Bieske, Rev. Sci. Instrum. **83**, (2012).
- [23] J. E. Sader and L. White, J. Appl. Phys. **74**, 1 (1993).
- [24] N. R. Shatil, M. E. Homer, L. Picco, P. G. Martin, and O. D. Payton, Appl. Phys. Lett. **110**, 223101 (2017).
- [25] J. E. Sader, James W. M. Chon, and P. Mulvaney, Rev. Sci. Instrum. **70**, 3967 (1999).
- [26] J. Lübbe, L. Doering, and M. Reichling, Meas. Sci. Technol. **23**, 45401 (2012).
- [27] C. A. Clifford and M. P. Seah, Nanotechnology **16**, 1666 (2005).
- [28] J. R. Lozano, D. Kiracofe, J. Melcher, R. Garcia, and A. Raman, Nanotechnology **21**, 465502 (2010).
- [29] J. W. M. Chon and P. Mulvaney, J. Appl. Phys. **87**, 3978 (2000).
- [30] C. P. Green, H. Lioe, J. P. Cleveland, R. Proksch, P. Mulvaney, and J. E. Sader, Rev. Sci. Instrum. **75**, 1988 (2004).
- [31] J. L. Hutter and J. Bechhoefer, Rev. Sci. Instrum. **64**, 1868 (1993).
- [32] H.-J. Butt and M. Jaschke, Nanotechnology **6**, 1 (1995).
- [33] J. E. Sader, J. Pacifico, C. P. Green, and P. Mulvaney, J. Appl. Phys. **97**, 124903 (2005).
- [34] A. F. Payam, Ultramicroscopy **135**, 84 (2013).

- [35] S. Basak, A. Raman, and S. V. Garimella, *J. Appl. Phys.* **99**, (2006).
- [36] R. C. Tung, J. P. Killgore, and D. C. Hurley, *Rev. Sci. Instrum.* **84**, 73703 (2013).
- [37] A. F. Payam and M. Fathipour, *Micron* **70**, 50 (2015).
- [38] A. Maali, C. Hurth, R. Boisgard, C. Jai, T. Cohen-Bouhacina, and J. Aimé, *J. Appl. Phys.* **97**, 74907 (2005).
- [39] A. F. Payam, W. Trewby, and K. Voitchovsky, *Analyst* **142**, 1492 (2017).
- [40] J. R. Lozano and R. Garcia, *Phys. Rev. B* **79**, 14110 (2009).
- [41] R. Garcia and E. T. Herruzo, *Nat. Nanotechnol.* **7**, 217 (2012).
- [42] C. A. Amo, A. P. Perrino, A. F. Payam, and R. Garcia, *ACS Nano* **11**, 8650 (2017).